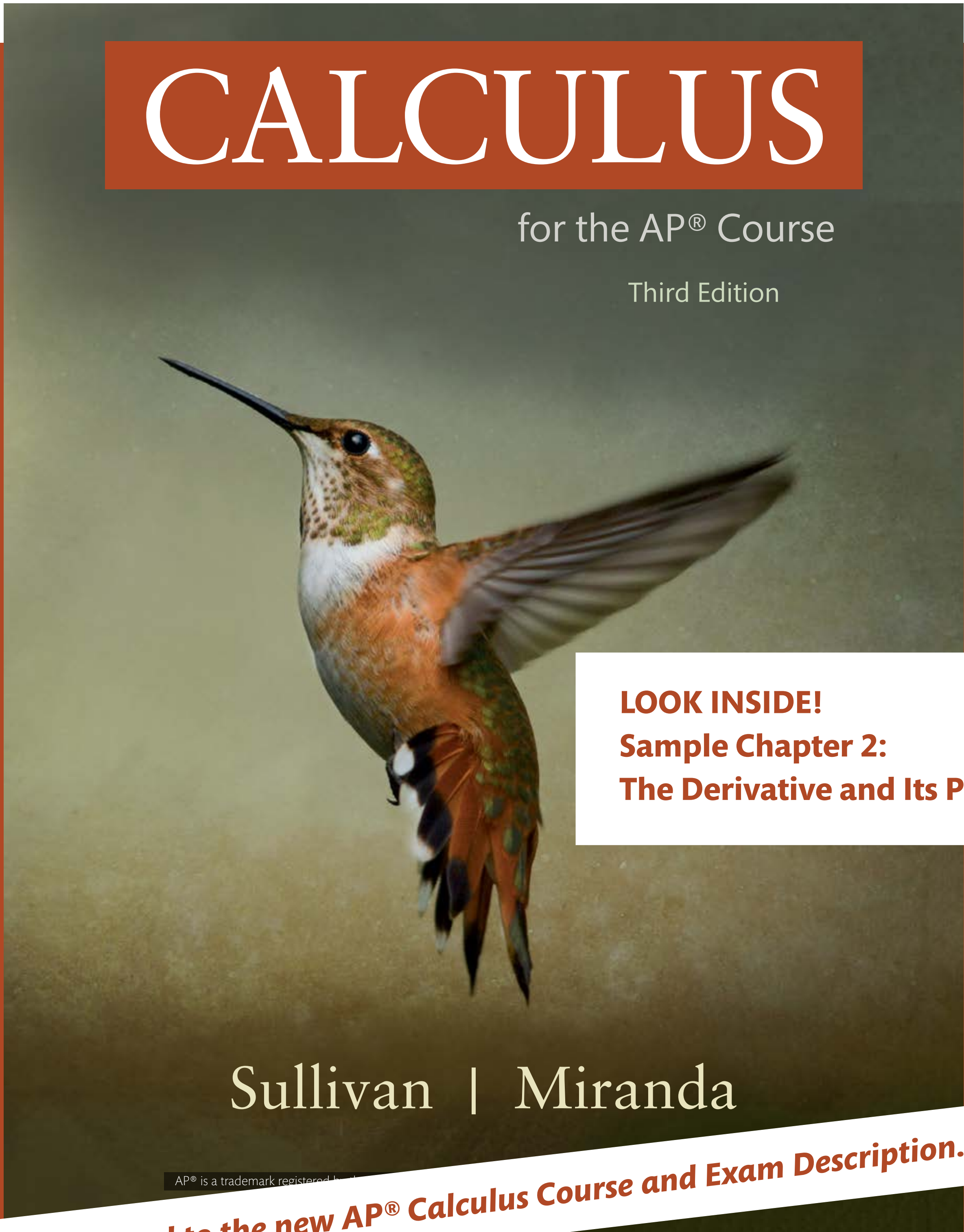
Teacher's Edition



LOOK INSIDE!
Sample Chapter 2:
The Derivative and Its Properties

Fully Aligned to the new AP® Calculus Course and Exam Description.

Joshua Newton   Brent Ferguson

Two of the most trusted authors in calculus.

Michael Sullivan

Michael Sullivan, Emeritus Professor of Mathematics at Chicago State University, received a Ph.D. in mathematics from the Illinois Institute of Technology. Before retiring, Mike taught at Chicago State for 35 years, where he honed an approach to teaching and writing that forms the foundation for his textbooks.

Mike has been writing for more than 35 years and currently has 15 books in print. Mike is a member of the American Mathematical Association of Two Year Colleges, the American Mathematical Society, the Mathematical Association of America, and the Textbook and Academic Authors Association. Mike serves on the governing board of TAA and represents TAA on the board of the Authors Coalition of America, a consortium of 22 author/creator organizations in the United States. In 2007, he received the TAA Lifetime Achievement Award.

His influence in the field of mathematics extends to his four children: Kathleen, who teaches college mathematics; Michael III, who also teaches college mathematics, and who is his coauthor on two precalculus series; Dan, who is a sales director for a college textbook publishing company; and Colleen, who teaches middle-school and secondary school mathematics. Twelve grandchildren round out the family. Mike would like to dedicate *Calculus for the AP® Course*, Third Edition, to his four children, 12 grandchildren, and future generations.

Kathleen Miranda

Kathleen Miranda, Ed.D from St. John's University, is an Emeritus Associate Professor of the State University of New York (SUNY), where she taught for 25 years. Kathleen is a recipient of the prestigious New York State Chancellor's Award for Excellence in Teaching, and she particularly enjoys teaching mathematics to underprepared and fearful students. Kathleen has served as director of Curriculum and Assessment Development at SUNY Old Westbury.

In addition to her extensive classroom experience, Kathleen has worked as accuracy reviewer and solutions author on several mathematics textbooks, including Michael Sullivan's *Brief Calculus* and *Finite Mathematics*. Kathleen's goal is to help students unlock the complexities of calculus and appreciate its many applications. Kathleen has four children: Edward, a plastic surgeon in San Francisco; James, an anesthesiologist in Philadelphia; Kathleen, a chemical engineer who directs a biologics division at a major pharmaceutical firm; and Michael, a corporate strategy specialist and entrepreneur. Kathleen would like to dedicate *Calculus for the AP® Course*, Third Edition, to her children and grandchildren.

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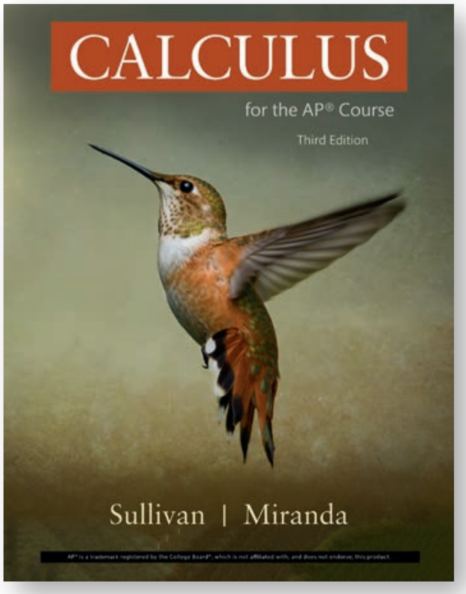

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This edition has been carefully developed to ensure that it adheres to the unit structure and coverage as set forth in the 2019 CED. Further, it aligns with the College Board's overarching structure, meaning every Big Idea, Mathematical Practice, and Student Skill. This edition also aligns with the revised pedagogy of Enduring Understanding, Learning Objective, and Essential Knowledge statement that flow from the three revised Big Ideas.

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COLLEGE BOARD CED	Sullivan and Miranda *Calculus for the AP® Course*, Third Edition	Recommendations: College Board Assessments
	Chapter P: Preparing for Calculus	
Unit 1: Limits and Continuity	**Unit 1: Limits and Continuity** Chapter 1: Limits and Continuity	*Assign Personal Progress Check #1*
Unit 2: Differentiation: Definition and Fundamental Properties	**Unit 2: Differentiation: Definition and Fundamental Properties** Chapter 2: The Derivative and Its Properties	*Assign Personal Progress Check #2*
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	**Unit 3: Differentiation: Composite, Implicit, and Inverse Functions** Chapter 3: The Derivative of Composite, Implicit, and Inverse Functions	*Assign Personal Progress Check #3*
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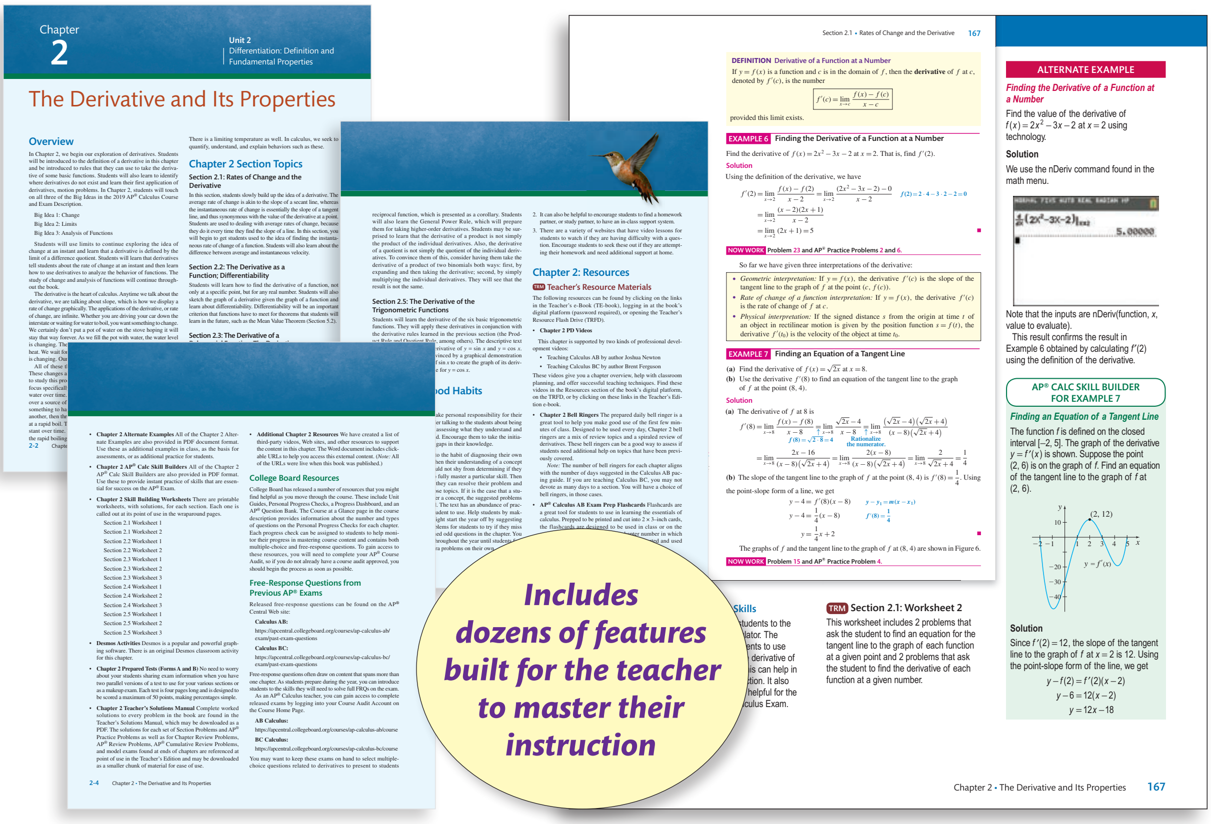

Written to help you realize success, the Teacher's Edition for Sullivan and Miranda, *Calculus for the AP® Course*, Third Edition, provides the built-in support you need, whether you are new to teaching AP ® Calculus or are a veteran teacher. Its goal is to make every teacher a confident, energized, and resourceful teacher even before the first day of class begins.

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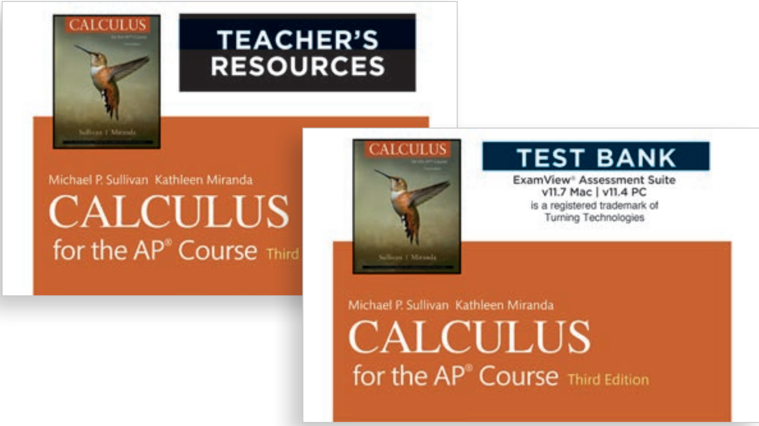

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Chapter

2

Unit 2

Differentiation: Definition and Fundamental Properties

The Derivative and Its Properties

Overview

In Chapter 2, we begin our exploration of derivatives. Students will be introduced to the definition of a derivative in this chapter and be introduced to rules that they can use to take the derivative of some basic functions. Students will also learn to identify where derivatives do not exist and learn their first application of derivatives, motion problems. In Chapter 2, students will touch on all three of the Big Ideas in the 2019 AP® Calculus Course and Exam Description.

Big Idea 1: Change

Big Idea 2: Limits

Big Idea 3: Analysis of Functions

Students will use limits to continue exploring the idea of change at an instant and learn that a derivative is defined by the limit of a difference quotient. Students will learn that derivatives tell students about the rate of change at an instant and then learn how to use derivatives to analyze the behavior of functions. The study of change and analysis of functions will continue throughout the book.

The derivative is the heart of calculus. Anytime we talk about the derivative, we are talking about slope, which is how we display a rate of change graphically. The applications of the derivative, or rate of change, are infinite. Whether you are driving your car down the interstate or waiting for water to boil, you want something to change. We certainly don't put a pot of water on the stove hoping it will stay that way forever. As we fill the pot with water, the water level is changing. Then we place the pot on the stove and turn up the heat. We wait for the temperature to change. While we wait, time is changing. Our position in the kitchen likely changes as well.

All of these things, plus infinitely more, happen every day. These changes also influence other changes. In calculus, we aim to study this process by focusing on one item at a time. We may focus specifically on the rate of change of the temperature of the water over time. As we watch a pot of water that we just placed over a source of heat, it takes what often seems like forever for something to happen. Finally, we observe a single bubble, then another, then three, and five, and before we know it, the water is at a rapid boil. The rate of change of the temperature is not constant over time. There is a long period of waiting that precedes the rapid boiling. Also, the water does not become infinitely hot. There is a limiting temperature as well. In calculus, we seek to quantify, understand, and explain behaviors such as these.

Chapter 2 Section Topics

Section 2.1: Rates of Change and the Derivative

In this section, students slowly build up the idea of a derivative. The average rate of change is akin to the slope of a secant line, whereas the instantaneous rate of change is essentially the slope of a tangent line, and thus synonymous with the value of the derivative at a point. Students are used to dealing with average rates of change, because they do it every time they find the slope of a line. In this section, you will begin to get students used to the idea of finding the instantaneous rate of change of a function. Students will also learn about the difference between average and instantaneous velocity.

Section 2.2: The Derivative as a Function; Differentiability

Students will learn how to find the derivative of a function, not only at a specific point, but for any real number. Students will also sketch the graph of a derivative given the graph of a function and learn about differentiability. Differentiability will be an important criterion that functions have to meet for theorems that students will learn in the future, such as the Mean Value Theorem (Section 5.2).

Section 2.3: The Derivative of a Polynomial Function; The Derivative of $y = e^x$

In this section, the students will begin to build knowledge of common derivatives. They will learn rules such as the Power Rule that will allow them to find the derivative of all polynomial functions without having to use the limit process. Students will also learn how to take the derivative of $y = e^x$ and other exponentials.

Section 2.4: Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

Students will learn to distinguish the Product Rule and Quotient Rule. Related to the Quotient Rule is the derivative of the

reciprocal function, which is presented as a corollary. Students will also learn the General Power Rule, which will prepare them for taking higher-order derivatives. Students may be surprised to learn that the derivative of a product is not simply the product of the individual derivatives. Also, the derivative of a quotient is not simply the quotient of the individual derivatives. To convince them of this, consider having them take the derivative of a product of two binomials both ways: first, by expanding and then taking the derivative; second, by simply multiplying the individual derivatives. They will see that the result is not the same.

Section 2.5: The Derivative of the Trigonometric Functions

Students will learn the derivative of the six basic trigonometric functions. They will apply these derivatives in conjunction with the derivative rules learned in the previous section (the Product Rule and Quotient Rule, among others). The descriptive text provides proofs for the derivative of $y = \sin x$ and $y = \cos x$. Many students will be convinced by a graphical demonstration of where to use the graph of $\sin x$ to create the graph of its derivative. The same can be done for $y = \cos x$.

Promoting Good Habits and Skills

Students should begin to take personal responsibility for their own understanding. Consider talking to the students about being honest with themselves in assessing what they understand and what they do not understand. Encourage them to take the initiative to do something about gaps in their knowledge.

1. Students have to get into the habit of diagnosing their own lacks and identifying when their understanding of a concept has flaws. Students should not shy from determining if they require more practice to fully master a particular skill. Then they have to plan how they can resolve their problem and exert more effort on those topics. If it is the case that a student can't seem to master a concept, the suggested problems should be supplemented. The text has an abundance of practice problems for the student to use. Help students by making suggestions. You might start the year off by suggesting some supplemental problems for students to try if they miss most or all of the assigned odd questions in the chapter. You can lessen this support throughout the year until students feel comfortable picking extra problems on their own.
2. It can also be helpful to encourage students to find a homework partner, or study partner, to have an in-class support system.
3. There are a variety of websites that have video lessons for students to watch if they are having difficulty with a question. Encourage students to seek these out if they are attempting their homework and need additional support at home.

Chapter 2: Resources

TRM Teacher's Resource Materials

The following resources can be found by clicking on the links in the Teacher's e-Book (TE-book), logging in at the book's digital platform (password required), or opening the Teacher's Resource Flash Drive (TRFD).

- **Chapter 2 PD Videos**

This chapter is supported by two kinds of professional development videos:

- Teaching Calculus AB by author Joshua Newton
- Teaching Calculus BC by author Brent Ferguson

These videos give you a chapter overview, help with classroom planning, and offer successful teaching techniques. Find these videos in the Resources section of the book's digital platform, on the TRFD, or by clicking on these links in the Teacher's Edition e-book.

- **Chapter 2 Bell Ringers** The prepared daily bell ringer is a great tool to help you make good use of the first few minutes of class. Designed to be used every day, Chapter 2 bell ringers are a mix of review topics and a spiraled review of derivatives. These bell ringers can be a good way to assess if students need additional help on topics that have been previously covered.

  Note: The number of bell ringers for each chapter aligns with the number of days suggested in the Calculus AB pacing guide. If you are teaching Calculus BC, you may not devote as many days to a section. You will have a choice of bell ringers, in those cases.
- **AP® Calculus AB Exam Prep Flashcards** Flashcards are a great tool for students to use in learning the essentials of calculus. Prepped to be printed and cut into 2 × 3–inch cards, the flashcards are designed to be used in class or on the go. Each card is labeled with the chapter number in which the topic is first introduced so they can be sorted and used throughout the course as a study aid.

- **Chapter 2 Alternate Examples** All of the Chapter 2 Alternate Examples are also provided in PDF document format. Use these as additional examples in class, as the basis for assessments, or as additional practice for students.
- **Chapter 2 AP® Calc Skill Builders** All of the Chapter 2 AP® Calc Skill Builders are also provided in PDF format. Use these to provide instant practice of skills that are essential for success on the AP® Exam.
- **Chapter 2 Skill Building Worksheets** There are printable worksheets, with solutions, for each section. Each one is called out at its point of use in the wraparound pages.

  Section 2.1 Worksheet 1
  Section 2.1 Worksheet 2
  Section 2.2 Worksheet 1
  Section 2.2 Worksheet 2
  Section 2.3 Worksheet 1
  Section 2.3 Worksheet 2
  Section 2.3 Worksheet 3
  Section 2.4 Worksheet 1
  Section 2.4 Worksheet 2
  Section 2.4 Worksheet 3
  Section 2.5 Worksheet 1
  Section 2.5 Worksheet 2
  Section 2.5 Worksheet 3
- **Desmos Activities** Desmos is a popular and powerful graphing software. There is an original Desmos classroom activity for this chapter.
- **Chapter 2 Prepared Tests (Forms A and B)** No need to worry about your students sharing exam information when you have two parallel versions of a test to use for your various sections or as a makeup exam. Each test is four pages long and is designed to be scored a maximum of 50 points, making percentages simple.
- **Chapter 2 Teacher's Solutions Manual** Complete worked solutions to every problem in the book are found in the Teacher's Solutions Manual, which may be downloaded as a PDF. The solutions for each set of Section Problems and AP® Practice Problems as well as for Chapter Review Problems, AP® Review Problems, AP® Cumulative Review Problems, and model exams found at ends of chapters are referenced at point of use in the Teacher's Edition and may be downloaded as a smaller chunk of material for ease of use.
- **Additional Chapter 2 Resources** We have created a list of third-party videos, Web sites, and other resources to support the content in this chapter. The Word document includes clickable URLs to help you access this external content. (*Note:* All of the URLs were live when this book was published.)

College Board Resources

College Board has released a number of resources that you might find helpful as you move through the course. These include Unit Guides, Personal Progress Checks, a Progress Dashboard, and an AP® Question Bank. The Course at a Glance page in the course description provides information about the number and types of questions on the Personal Progress Checks for each chapter. Each progress check can be assigned to students to help monitor their progress in mastering course content and contains both multiple-choice and free-response questions. To gain access to these resources, you will need to complete your AP® Course Audit, so if you do not already have a course audit approved, you should begin the process as soon as possible.

Free-Response Questions from Previous AP® Exams

Released free-response questions can be found on the AP® Central Web site:

Calculus AB:

https://apcentral.collegeboard.org/courses/ap-calculus-ab/exam/past-exam-questions

Calculus BC:

https://apcentral.collegeboard.org/courses/ap-calculus-bc/exam/past-exam-questions

Free-response questions often draw on content that spans more than one chapter. As students prepare during the year, you can introduce students to the skills they will need to solve full FRQs on the exam.

As an AP® Calculus teacher, you can gain access to complete released exams by logging into your Course Audit Account on the Course Home Page.

AB Calculus:

https://apcentral.collegeboard.org/courses/ap-calculus-ab/course

BC Calculus:

https://apcentral.collegeboard.org/courses/ap-calculus-bc/course

You may want to keep these exams on hand to select multiple-choice questions related to derivatives to present to students for further practice. Likewise, you may want to make a chart or find some other organizing tool that allows you to quickly group multiple-choice questions by topic so they may be used at appropriate times throughout the year.

Here are some free-response questions that you may use in conjunction with this chapter:

Year	Question	Topic
2011	1a	Particle motion: Is speed increasing or decreasing?
2011	2a	Derivative: Estimating the derivative given data
2011	3a	Tangent line: Write the equation of a tangent line.
2011	6a, 6b	Continuity and derivative of a piecewise function
2012	1a	Derivative: Estimating the derivative given data
2012	4a, 4b, 4c	Derivative, tangent line, continuity
2012	6a	Particle motion: When is the particle moving to the left?
2013	1a	Derivative and interpretation
2013	2a, 2c, 2d	Particle motion: Find where speed = 2; change directions; speed increasing or decreasing?
2013	3a	Derivative: Estimating the derivative given data
2014	1a, 1b	Average rate of change, derivative and interpretation
2014	4a, 4b	Derivative: Estimating the derivative given data Intermediate Value Theorem
2015	3a, 3c, 3d	Derivative: Estimating the derivative given data Derivative from a function, average rate of change
2016	1a	Derivative: Estimating the derivative given data
2016	2a, 2b	Particle motion: Speeding up or slowing down, change directions
2017	2b	Derivative: Derivative on a calculator with explanation
2018	2a, 2d	Particle motion: Find acceleration, find where velocities are equal.
2018	4a	Derivative: Estimating the derivative given data
2018	5a, 5b	Average rate of change Product rule, tangent line
2019	3a	Limit using derivative found from graph
2019	6a, 6b	Derivative using product rule

You may also want to direct your students to the following Khan Academy site. The site has videos with explanations about how to solve free-response questions from select years:

https://www.khanacademy.org/math/calculus-home/ap-calc-topic

Chapter 2: Pacing Guides, Objectives, and Suggested Assignments

These pacing guides are based on a schedule with 125 standard 45-minute classroom sessions before the exam. This 125-day course includes assessment days and allows about 3 weeks for review before the AP® Calculus exam.

The suggested homework assignments list odd-numbered problems whenever possible, so students can check their answers against the back-of-the-book answers. If you would rather students not have access to the answers while doing homework, adding 1 to the exercise numbers usually will do the trick, because the homework problems typically are paired. The answers for the even-numbered problems are not in the Answer appendix.

The authors observe pairing strictly in the Skill Building category. The Applications and Extensions usually are paired, although more advanced problems tend not to be paired. The AP® Practice Problems are not paired.

Calculus AB Pacing Guide

Day	Topic	Sullivan/Miranda Chapter Objectives	Suggested Assignment
1	Section 2.1	1. Find equations for the tangent line and the normal line to the graph of a function 2. Find the rate of change of a function 3. Find average velocity and instantaneous velocity 4. Find the derivative of a function at a number	7, 11, 15, 17, 23, 27, 31, 33, 43, 51
2	Section 2.1	1. Find equations for the tangent line and the normal line to the graph of a function 2. Find the rate of change of a function 3. Find average velocity and instantaneous velocity 4. Find the derivative of a function at a number	All AP® Practice Problems
3	Section 2.2	1. Define the derivative function 2. Graph the derivative function 3. Identify where a function is not differentiable	13, 15, 19, 23, 25, 27, 29, 31–34, 35, 39, 41, 43, 45, 67
4	Section 2.2	1. Define the derivative function 2. Graph the derivative function 3. Identify where a function is not differentiable	All AP® Practice Problems
5	Section 2.3	1. Differentiate a constant function 2. Differentiate a power function 3. Differentiate the sum and difference of two functions 4. Differentiate the exponential function $y = e^x$	1, 7–37 (odd)
6	Section 2.3	1. Differentiate a constant function 2. Differentiate a power function 3. Differentiate the sum and difference of two functions 4. Differentiate the exponential function $y = e^x$	43, 47, and all AP® Practice Problems
7	Section 2.4	1. Differentiate the product of two functions 2. Differentiate the quotient of two functions 3. Find higher-order derivatives 4. Find the acceleration of an object in rectilinear motion	9–15 (odd), 19, 23, 25, 31, 33, 37
8	Section 2.4	1. Differentiate the product of two functions 2. Differentiate the quotient of two functions 3. Find higher-order derivatives 4. Find the acceleration of an object in rectilinear motion	41, 45, 49, 53, 57, 67, 73, 81, 83, 91, and all AP® Practice Problems
9	Section 2.5	1. Differentiate trigonometric functions	5–23 (odd), 29, 33, 35, 39–45 (odd)
10	Section 2.5	1. Differentiate trigonometric functions	55, 57, 65, and all AP® Practice Problems
11	Review		Chapter 2 Review Exercises: 1, 5, 11, 19, 27, 31, 41, 45, 49, 53, 59, 67, 71
12	Review		AP® Review Problems: Chapter 2
13	Test		AP® Cumulative Review Problems: Chapters 1–2

Chapter
2

Calculus BC Pacing Guide

Day	Topic	Sullivan/Miranda Chapter Objectives	Suggested Assignment
1	Section 2.1	1. Find equations for the tangent line and the normal line to the graph of a function 2. Find the rate of change of a function 3. Find average velocity and instantaneous velocity 4. Find the derivative of a function at a number	11, 15, 17, 19, 23, 25, 31, 33, 37, 39, 41, 43, 47, 51, 55, and all AP® Practice Problems
2	Section 2.2	1. Define the derivative function 2. Graph the derivative function 3. Identify where a function is not differentiable	5, 9, 15, 19, 20, 25, 29–34, 35–49 (odd), 57, 61
3	Section 2.2	1. Define the derivative function 2. Graph the derivative function 3. Identify where a function is not differentiable	65, 67, 69, 76, 77, 78, and all AP® Practice Problems
4	Section 2.3	1. Differentiate a constant function 2. Differentiate a power function 3. Differentiate the sum and difference of two functions 4. Differentiate the exponential function $y = e^x$	1, 7, 13, 19, 21, 25, 31, 33–47 (odd), 53, 61, 63, 65, 71, 76, 77, 85, and all AP® Practice Problems
5	Section 2.4	1. Differentiate the product of two functions 2. Differentiate the quotient of two functions 3. Find higher-order derivatives 4. Find the acceleration of an object in rectilinear motion	9, 13, 23, 25, 31, 37, 41, 45, 47, 55, 59, 61, 65, 67, 69, 77, 83, 91
6	Section 2.4	1. Differentiate the product of two functions 2. Differentiate the quotient of two functions 3. Find higher-order derivatives 4. Find the acceleration of an object in rectilinear motion	81, 85, 87, 99, 108, and all AP® Practice Problems
7	Section 2.5	1. Differentiate trigonometric functions	1–5, 13, 15, 23, 27, 29, 31, 35, 45, 51, 55, 57, 61, 63, 65, 82, and all AP® Practice Problems
8	Review		Chapter 2 Review Exercises: 1, 11, 14–29, 47–53, 65, 67, and all AP® Practice Problems
9	Test		Read 3.1–3.4 after your test to prepare for classes ahead

Relationship to the AP® Calculus Curriculum Framework

Chapter 2 aligns with Unit 2 of the fall 2019 updated course description. Chapter 2 addresses College Board's new requirements listed in the updated course description in the following ways:

AP® Calculus: The Mathematical Practices and Suggested Skills

The 2019 course framework brings Mathematical Practices to the fore. There are now four Mathematical Practices.

Mathematical Practices in Use in Chapter 2

- Practice 1: Implementing Mathematical Processes (Sections 2.2, 2.3, 2.4, and 2.5)
- Practice 2: Connecting Representations (Section 2.1)
- Practice 3: Justification (Section 2.2)
- Practice 4: Communication and Notation (Section 2.2)

In combination with the Mathematical Practices, Chapter 2 develops many of the AP® Calculus suggested skills for students. The suggested skills applicable by section are listed in the table on the facing page by their alphanumeric code.

Suggested Skills in Use in Chapter 1

- Practice 1: Implementing Mathematical Processes (1.D and 1.E)
- Practice 2: Connecting Representations (2.B)
- Practice 4: Communication and Notation (4.C)

The Big Ideas for the AP® Course

Course content is grounded in the Big Ideas, which are cross-cutting concepts that appear throughout the course. There are now a total of three Big Ideas in AP® Calculus. Chapter 2 focuses mainly on two of the Big Ideas outlined in the course framework:

- **Big Idea 1: Change (CHA)** Students will begin to learn about the rate at which a function changes at an instant. This is a new concept for students mathematically. In the past, they have always found an average rate of change of a function, such as a line. Derivatives allow us to think about how a quantity is changing at a specific point in time.
- **Big Idea 3: Analysis of Functions (FUN)** Students will begin to use derivatives to identify information about functions. Derivatives can help us learn about how a function behaves, and they allow us to answer questions. We will also explore how to interpret what a derivative tells us about problems in context.

Chapter 2

Chapter Coverage Related to Suggested Skills, Learning Objectives, and Essential Knowledge

Section	Title	AP® Calculus Suggested Skills	AP® Calculus Learning Objectives	AP® Calculus Essential Knowledge
2.1	Rates of Change and the Derivative	2.B	CHA-2.A; CHA-2.B	CHA-2.A.1; CHA-2.B.1
2.2	The Derivative as a Function; Differentiability	1.D	CHA-2.B; CHA-2.C; CHA-2.D; FUN-2.A	CHA-2.B.2; CHA-2.B.3; CHA-2.B.4; CHA-2.C.1; CHA-2.D.1; CHA-2.D.2; FUN-2.A.1; FUN-2.A.2
2.3	The Derivative of a Polynomial Function; The Derivative of $y = e^x$	4.C	FUN-3.A	FUN-3.A.1; FUN-3.A.2; FUN-3.A.3; FUN-3.A.4
2.4	Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives	1.E	FUN-3.B	FUN-3.B.1; FUN-3.B.2
2.5	The Derivative of Trigonometric Functions		FUN-3.A; FUN-3.B; LIM-3.A	FUN-3.A.4; FUN-3.B.3; LIM-3.A.1

PD Chapter 2 PD Videos

Chapter 2 is supported by two kinds of professional development videos:

- Teaching Calculus AB by author Joshua Newton
- Teaching Calculus BC by author Brent Ferguson

These videos give you a chapter overview, help with classroom planning, and offer successful teaching techniques. Find these videos in the Resources section of the book's digital platform, on the TRFD, or by clicking on these links in the Teacher's Edition e-book.

This chapter opens by giving the students some background information on the lunar module, the spacecraft that enabled the astronauts to land on the Moon. The project at the end of the chapter walks the students through the physics that describes the trajectory specified by the engineers who made the trip possible. Students will use what they have learned in this chapter to model the motion of this spacecraft.

TRM Chapter 2 Bell Ringers

Bell ringers for Chapter 2 help students review topics from Chapter 1, prepare students for lessons by helping them recall prior knowledge, and include spiraled review of derivative topics. The review of older topics is targeted to skills that students will need to be successful throughout the chapter. There is a bell ringer for each instructional day and one for the day that you review the chapter.

TRM AP® Calc AB Exam Prep Flashcards

You may want to give your students the AP® Calc AB Exam Prep Flashcards now so that they may use them throughout the year. Have them extract the Chapter 2 cards by referencing the code in the bottom corner.

Where to Find the Teacher Resources?

All of the Teacher Resource Materials listed in the blue pages for this chapter and referenced through the icons may be found by clicking on the links in the Teacher's e-Book (TE-book), logging into LaunchPad (password required) go.bfwpub.com/APCalculus3e, or opening the Teacher's Resource Flash Drive (TRFD).

Chapter 2 The Derivative and Its Properties

Unit 2
Differentiation: Definition and Fundamental Properties

Rolls Press/Popperfoto/Getty Images

The *Apollo* Lunar Module

"One Giant Leap for Mankind"

On May 25, 1961, in a special address to Congress, U.S. President John F. Kennedy proposed the goal "before this decade is out, of landing a man on the Moon and returning him safely to the Earth." Roughly eight years later, on July 16, 1969, a Saturn V rocket launched from the Kennedy Space Center in Florida, carrying the *Apollo 11* spacecraft and three astronauts—Neil Armstrong, Buzz Aldrin, and Michael Collins—bound for the Moon.

The *Apollo* spacecraft had three parts: the Command Module with a cabin for the three astronauts; the Service Module that supported the Command Module with propulsion, electrical power, oxygen, and water; and the Lunar Module for landing on the Moon. After its launch, the spacecraft traveled for three days until it entered into lunar orbit. Armstrong and Aldrin then moved into the Lunar Module, which they landed in the flat expanse of the Sea of Tranquility. After more than 21 hours, the first humans to touch the surface of the Moon crawled into the Lunar Module and lifted off to rejoin the Command Module, which Collins had been piloting in lunar orbit. The three astronauts then headed back to Earth, where they splashed down in the Pacific Ocean on July 24.

Explore some of the physics at work that allowed engineers and pilots to successfully maneuver the Lunar Module to the Moon's surface in the Chapter 2 Project on page 215.

Chapter 2 opens by returning to the tangent problem to find an equation of the tangent line to the graph of a function f at a point $P=(c, f(c))$. Remember in Section 1.1 we found that the slope of a tangent line was a limit,

$$m_{\tan}=\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

This limit turns out to be one of the most significant ideas in calculus, the *derivative*.

In this chapter, we introduce interpretations of the derivative, treat the derivative as a function, and consider some properties of the derivative. By the end of the chapter, you will have a collection of basic derivative formulas and derivative rules that will be used throughout your study of calculus.

AP® EXAM TIP

The derivative is an important concept in the AP® Calculus curriculum.

2.1 Rates of Change and the Derivative

OBJECTIVES *When you finish this section, you should be able to:*

1. **Find equations for the tangent line and the normal line to the graph of a function (p. 162)**
2. **Find the rate of change of a function (p. 163)**
3. **Find average velocity and instantaneous velocity (p. 164)**
4. **Find the derivative of a function at a number (p. 166)**

In Chapter 1, we discussed the tangent problem: *Given a function f and a point P on its graph, what is the slope of the tangent line to the graph of f at P*? See Figure 1, where ℓ_T is the tangent line to the graph of f at the point $P=(c, f(c))$.

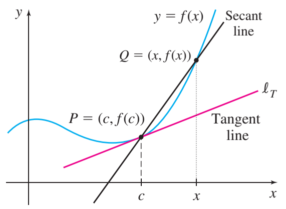


Figure 1 $m_{\sec}$ = slope of the secant line.

The tangent line ℓ_T to the graph of f at P must contain the point P. Since finding the slope requires two points, and we have only one point on the tangent line ℓ_T, we reason as follows.

Suppose we choose any point $Q=(x, f(x))$, other than P, on the graph of f. (Q can be to the left or to the right of P; we chose Q to be to the right of P.) The line containing the points $P=(c, f(c))$ and $Q=(x, f(x))$ is a secant line of the graph of f. The slope $m_{\sec}$ of this secant line is

$$m_{\sec}=\frac{f(x)-f(c)}{x-c} \qquad (1)$$

Figure 2 shows three different points Q_1, Q_2, and Q_3 on the graph of f that are successively closer to the point P, and three associated secant lines ℓ_1, ℓ_2, and ℓ_3. The closer the points Q are to the point P, the closer the secant lines are to the tangent line ℓ_T. The line ℓ_T, the *limiting position* of these secant lines, is the *tangent line to the graph* of f at P.

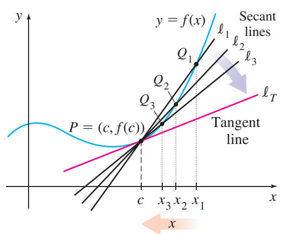


Figure 2 $m_{\tan}=\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$

If the limiting position of the secant lines is the tangent line, then the limit of the slopes of the secant lines should equal the slope of the tangent line. Notice in Figure 2 that as the points Q_1, Q_2, and Q_3 move closer to the point P, the numbers x get closer to c. So, equation (1) suggests that

$$\begin{aligned} m_{\tan} &= \text{Slope of the tangent line to } f \text{ at } P \\ &= \text{Limit of } \frac{f(x)-f(c)}{x-c} \text{ as } x \text{ gets closer to } c \\ &= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \end{aligned}$$

provided the limit exists.

TEACHING THE AP® TIP

Derivatives are foundational for all of calculus. Students will use them to analyze functions and solve related rates and optimization problems. Students will reverse the process to find antiderivatives when solving problems involving integrals and differential equations. Calculus BC students will use derivatives to create Taylor Polynomials. Ensure that students know that the skills they learn in Chapters 2 and 3 will be used throughout the course.

BIG IDEAS TIP

Big Idea 1: Change

In this chapter, students will begin to think about the rate at which change is happening at a moment. In previous courses, students have thought mainly about the average rate of change as opposed to an instantaneous rate of change. This change (in how we're thinking about change) can be difficult for students at first but is critical for their understanding of calculus. As you introduce the idea of a derivative at a number, talk to students about what it means for a function to be changing. It can be helpful to include some real-world examples of change at an instant: The speedometer in your car tells you your speed at an exact instant, or a stadium manager might want to know the rate at which fans are arriving 30 minutes before a game starts to make sure enough entrances are open to accommodate them.

Teaching Tip

Remind the students of the definition of slope that they are familiar with when naming the two points (x_1, y_1) and (x_2, y_2):

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Ask the students to compare this to the formula for the slope of a secant line.

$$m_{\sec}=\frac{f(x)-f(c)}{x-c}$$

The only difference between these two formulas is the way the two points are named. In this case, the points are named $(c, f(c))$ and $(x, f(x))$.

Then, point out that as point $(x, f(x))$ gets closer to the point $(c, f(c))$ (x approaches c), the slope of the secant line approaches the slope of the tangent line. Hence the formula:

$$m_{\tan}=\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

TRM Alternate Examples Section 2.1

You can find the Alternate Examples for this section in PDF format in the Teacher's Resource Materials.

TRM AP® Calc Skill Builders Section 2.1

You can find the AP® Calc Skill Builders for this section in PDF format in the Teacher's Resource Materials.

Teaching Tip

It is possible to teach this section in conjunction with Section 2.2. In this section, the students learn the definition of a derivative. They continue to use this method of differentiation as they learn about instantaneous rate of change, tangent lines, and normal lines. If you are pressed for time, the concepts of instantaneous rate of change, tangent lines, and normal lines can be expedited by addressing them once the students can take derivatives using the Power Rule in Section 2.3.

DEFINITION Tangent Line

The **tangent line** to the graph of f at a point P is the line containing the point $P=(c, f(c))$ and having the slope

$$m_{\tan}=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \tag{2}$$

provided the limit exists.

NOTE It is possible that the limit in (2) does not exist. The geometric significance of this is discussed in the next section.

The limit in equation (2) that defines the slope of the tangent line occurs so frequently that it is given a special notation $f'(c)$, read, "f prime of c," and called **prime notation**:

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \tag{3}$$

1 Find Equations for the Tangent Line and the Normal Line to the Graph of a Function

THEOREM Equation of a Tangent Line

If $m_{\tan}=f'(c)$ exists, then an equation of the tangent line to the graph of a function $y=f(x)$ at the point $P=(c, f(c))$ is

$$y-f(c)=f'(c)(x-c)$$

RECALL Two lines, neither of which is horizontal, with slopes m_1 and m_2, respectively, are perpendicular if and only if

$$m_1=-\frac{1}{m_2}$$

The line perpendicular to the tangent line at a point P on the graph of a function f is called the **normal line** to the graph of f at P.

An equation of the normal line to the graph of a function $y=f(x)$ at the point $P=(c, f(c))$ is

$$y-f(c)=-\frac{1}{f'(c)}(x-c)$$

provided $f'(c)$ exists and is not equal to zero. If $f'(c)=0$, the tangent line is horizontal, the normal line is vertical, and the equation of the normal line is $x=c$.

AP® EXAM TIP

Problems on the exam often ask about the tangent line and the normal line.

CALC CLIP

EXAMPLE 1 Finding Equations for the Tangent Line and the Normal Line

(a) Find the slope of the tangent line to the graph of $f(x)=x^2$ at the point $(-2, 4)$.

(b) Use the result from (a) to find an equation of the tangent line at the point $(-2, 4)$.

(c) Find an equation of the normal line to the graph of f at the point $(-2, 4)$.

(d) Graph f, the tangent line to f at $(-2, 4)$, and the normal line to f at $(-2, 4)$ on the same set of axes.

RECALL One way to find the limit of a quotient when the limit of the denominator is 0 is to factor the numerator and divide out common factors.

Solution

(a) At the point $(-2, 4)$, the slope of the tangent line is

$$f'(-2)=\lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to -2}\frac{x^2-(-2)^2}{x+2}=\lim_{x\to -2}\frac{x^2-4}{x+2}$$

$$=\lim_{x\to -2}(x-2)=-4$$

AP® CALC SKILL BUILDER FOR EXAMPLE 1

Finding Equations of the Tangent and the Normal Line

Find the slope of the tangent line to the graph of $f(x)=x^3$ at $c=1$. Write the equation of the tangent and normal line to the graph of f at the point (1, 1).

Solution

At $c=1$, the slope of the tangent line is

$$\begin{aligned} f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{x^3-1}{x-1} \\ &= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)} \\ &= \lim_{x\to 1}(x^2+x+1) \\ &= 3 \end{aligned}$$

Tangent line:

$$y-1=3(x-1)$$
$$y=3x-2$$

Normal line:

$$y-1=-\frac{1}{3}(x-1)$$
$$y=-\frac{1}{3}x+\frac{4}{3}$$

TEACHING THE AP® TIP

On the AP® Calculus Exam, students should be comfortable writing the equation of a line in either point-slope or slope-intercept form. On free-response questions, students can leave the equation of a tangent or normal line in any form. Get students in the habit of leaving lines in point-slope form on free-response questions and encourage students to not rewrite them in slope-intercept form. Rewriting the equation of a line adds more steps where a student might make a mistake and lose a point. On the multiple-choice, answers may be written in slope-intercept form, so following this practice in your class can give students a chance to practice simplification on multiple-choice questions.

Teaching Tip

It may help to show the students that the equation of the line $y-f(c)=f'(c)(x-c)$ is of the same form as the familiar point-slope form of a line: $y-y_1=m(x-x_1)$. Because many students are more familiar with the slope-intercept form than they are with the point-slope form, a brief review is worthwhile, since the tangent and normal lines are so much easier and quicker to write in point-slope form.

Later, an explicit connection will be made to yet another form: $y=y_1+m(x-x_1)$ in the Fundamental Theorem of Calculus.

Teaching Tip

Students may wonder why, in calculus, we stop saying "perpendicular" and start using a specific connotation of "normal" as well as a word likely to be new, "orthogonal." The word "perpendicular" describes a 2-dimensional relationship of lines and tends to be limited. "Normal" comes from the Latin *normalis*, meaning "made according to the carpenter's square," another 2-dimensional measure, although normal can refer to a point or a surface. For these reasons, the new term "orthogonal" starts to take on importance in describing 3-dimensional relationships.

NEED TO REVIEW? The point-slope form of a line is discussed in Appendix A.3, p. A19.

(b) We use the result from (a) and the point-slope form of an equation of a line to obtain an equation of the tangent line. An equation of the tangent line containing the point $(-2, 4)$ is

$$y - 4 = f'(-2)[x - (-2)] \quad \text{Point-slope form of an equation of the tangent line.}$$
$$y - 4 = -4 \cdot (x + 2) \quad f(-2) = 4; \quad f'(-2) = -4$$
$$y = -4x - 4 \quad \text{Simplify.}$$

(c) Since the slope of the tangent line to f at $(-2, 4)$ is -4, the slope of the normal line to f at $(-2, 4)$ is $\frac{1}{4}$.

Using the point-slope form of an equation of a line, an equation of the normal line is

$$y - 4 = \frac{1}{4}(x + 2)$$
$$y = \frac{1}{4}x + \frac{9}{2}$$

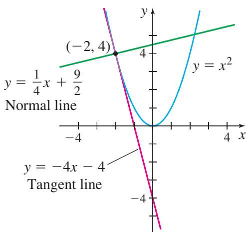


Figure 3 $f(x) = x^2$

(d) The graphs of f, the tangent line to the graph of f at the point $(-2, 4)$, and the normal line to the graph of f at $(-2, 4)$ are shown in Figure 3. ■

NOW WORK Problem 11 and AP® Practice Problems 1 and 5.

2 Find the Rate of Change of a Function

Everything in nature changes. Examples include climate change, change in the phases of the Moon, and change in populations. To describe natural processes mathematically, the ideas of change and rate of change are often used.

Recall that the average rate of change of a function $y = f(x)$ from c to x is given by

$$\text{Average rate of change} = \frac{f(x) - f(c)}{x - c} \qquad x \neq c$$

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f at c is the limit as x approaches c of the average rate of change. Symbolically, the instantaneous rate of change of f at c is

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

IN WORDS

- An average rate of change describes behavior over an interval.
- An instantaneous rate of change describes behavior at a number.

The expression "instantaneous rate of change" is often shortened to *rate of change*. Using prime notation, the **rate of change** of f at c is $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

EXAMPLE 2 Finding a Rate of Change

Find the rate of change of the function $f(x) = x^2 - 5x$ at:

(a) $c = 2$

(b) Any real number c

Solution

(a) For $c = 2$,

$$f(x) = x^2 - 5x \qquad \text{and} \qquad f(2) = 2^2 - 5 \cdot 2 = -6$$

Teaching Tip

Consider having the students each draw the parabola $y = x^2$. Then ask each of them to determine what values of x will yield a tangent line with a negative slope, a slope of zero, and a positive slope. Repeat this process for $y = x^3$. Discuss.

SUGGESTED SKILL 2.B

Having students draw the graph of a function as well as the tangent and normal lines they have found can help reinforce the idea that different representations of a function are connected. Graphing a function and its tangent line can help students see a connection between the analytic (algebraic) work they did and the graphical representation of the function.

AP® CALC SKILL BUILDER FOR EXAMPLE 2

Finding a Rate of Change

Find the rate of change of the function $f(x) = \sqrt{x}$ at $c = 4$.

Solution

The rate of change of f at $c = 4$ is

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$
$$= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$
$$= \lim_{x \to 4} \frac{(x - 4)}{(x - 4)(\sqrt{x} + 2)}$$
$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$
$$= \frac{1}{4}$$

Teaching Tip

Consider sharing this analogy with the students: The average rate of change is like a speed trap that a police officer might set up on the road. The instantaneous rate of change would be equivalent to the police officer using a radar gun. Suppose the speed trap consisted of two lines painted across the road. If there was enough distance between them, and the driver realized that she was going too fast, she would be able to slam on the breaks and decrease her average speed before crossing the second line. As the distance between the two lines decreased, however, the driver's average rate of change would approach that of her instantaneous rate of change at the first line.

The rate of change of f at $c=2$ is

$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{(x^2-5x)-(-6)}{x-2}=\lim_{x\to 2}\frac{x^2-5x+6}{x-2}$$
$$=\lim_{x\to 2}\frac{(x-2)(x-3)}{x-2}=\lim_{x\to 2}(x-3)=-1$$

(b) If c is any real number, then $f(c)=c^2-5c$, and the rate of change of f at c is

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{(x^2-5x)-(c^2-5c)}{x-c}=\lim_{x\to c}\frac{(x^2-c^2)-5(x-c)}{x-c}$$
$$=\lim_{x\to c}\frac{(x-c)(x+c)-5(x-c)}{x-c}=\lim_{x\to c}\frac{(x-c)(x+c-5)}{x-c}$$
$$=\lim_{x\to c}(x+c-5)=2c-5$$

■

NOW WORK Problem 17 and AP® Practice Problem 3.

EXAMPLE 3 Finding the Rate of Change in a Biology Experiment

In a metabolic experiment, the mass M of glucose decreases according to the function

$$M(t)=4.5-0.03t^2$$

where M is measured in grams (g) and t is the time in hours (h). Find the reaction rate $M'(t)$ at $t=1$ h.

Solution

The reaction rate at $t=1$ is $M'(1)$.

$$M'(1)=\lim_{t\to 1}\frac{M(t)-M(1)}{t-1}=\lim_{t\to 1}\frac{(4.5-0.03t^2)-(4.5-0.03)}{t-1}$$
$$=\lim_{t\to 1}\frac{-0.03t^2+0.03}{t-1}=\lim_{t\to 1}\frac{(-0.03)(t^2-1)}{t-1}=\lim_{t\to 1}\frac{(-0.03)(t-1)(t+1)}{t-1}$$
$$=-0.03\cdot 2=-0.06$$

The reaction rate at $t=1$ h is -0.06 g/h. That is, the mass M of glucose at $t=1$ h is decreasing at the rate of 0.06 g/h. ■

NOW WORK Problem 43.

3 Find Average Velocity and Instantaneous Velocity

Average velocity is a physical example of an average rate of change. For example, consider an object moving along a horizontal line with the positive direction to the right, or moving along a vertical line with the positive direction upward. Motion along a line is referred to as **rectilinear motion**. The object's location at time $t=0$ is called its **initial position**. The initial position is usually marked as the origin O on the line. See Figure 4. We assume the position s at time t of the object from the origin is given by a function $s=f(t)$. Here s is the signed, or directed, distance (using some measure of distance such as centimeters, meters, feet, etc.) of the object from O at time t (in seconds or hours). The function f is usually called the **position function** of the object.

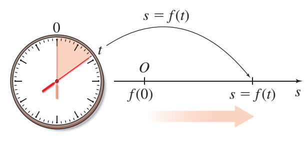


Figure 4 t is the travel time. s is the signed distance of the object from the origin at time t.

DEFINITION Average Velocity

The signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s=f(t)$. If at time t_0 the object is at $s_0=f(t_0)$ and at time t_1 the object is at $s_1=f(t_1)$, then the change in time is $\Delta t=t_1-t_0$ and the change in position is $\Delta s=s_1-s_0=f(t_1)-f(t_0)$. The average rate of change of position with respect to time is

$$\frac{\Delta s}{\Delta t}=\frac{f(t_1)-f(t_0)}{t_1-t_0}\qquad t_1\neq t_0$$

and is called the **average velocity** of the object over the interval $[t_0, t_1]$. See Figure 5.

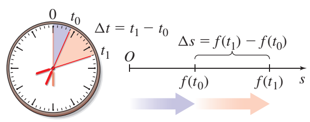


Figure 5 The average velocity is $\frac{\Delta s}{\Delta t}$.

EXAMPLE 4 Finding Average Velocity

Scott Prokop/Shutterstock

The Mike O'Callaghan–Pat Tillman Memorial Bridge spanning the Colorado River opened on October 16, 2010. Having a span of 1900 ft, it is the longest arch bridge in the Western Hemisphere, and its roadway is 890 ft above the Colorado River.

If a rock falls from the roadway, the function $s = f(t) = 16t^2$ gives the distance s, in feet, that the rock falls after t seconds for $0 \le t \le 7.458$. Here 7.458 s is the approximate time it takes the rock to fall 890 ft into the river. The average velocity of the rock during its fall is

$$\frac{\Delta s}{\Delta t} = \frac{f(7.458) - f(0)}{7.458 - 0} = \frac{890 - 0}{7.458} \approx 119.335 \text{ ft/s}$$ ■

NOW WORK AP® Practice Problem 7.

NOTE Here the motion occurs along a vertical line with the positive direction downward.

The average velocity of the rock in Example 4 approximates the average velocity over the interval [0, 7.458]. But the average velocity does not tell us about the velocity at any particular instant of time. That is, it gives no information about the rock's *instantaneous velocity*.

We can investigate the instantaneous velocity of the rock, say, at $t = 3$ s, by computing average velocities for short intervals of time beginning at $t = 3$. First we compute the average velocity for the interval beginning at $t = 3$ and ending at $t = 3.5$. The corresponding distances the rock has fallen are

$$f(3) = 16 \cdot 3^2 = 144 \text{ ft} \qquad \text{and} \qquad f(3.5) = 16 \cdot 3.5^2 = 196 \text{ ft}$$

Then $\Delta t = 3.5 - 3.0 = 0.5$, and during this 0.5-s interval,

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{f(3.5) - f(3)}{3.5 - 3} = \frac{196 - 144}{0.5} = 104 \text{ ft/s}$$

Table 1 shows average velocities of the rock for smaller intervals of time.

TABLE 1

Time interval	Start $t_0 = 3$	End t	Δt	$\frac{\Delta s}{\Delta t} = \frac{f(t) - f(t_0)}{t - t_0} = \frac{16t^2 - 144}{t - 3}$
[3, 3.1]	3	3.1	0.1	$\frac{\Delta s}{\Delta t} = \frac{f(3.1) - f(3)}{3.1 - 3} = \frac{16 \cdot 3.1^2 - 144}{0.1} = 97.6$
[3, 3.01]	3	3.01	0.01	$\frac{\Delta s}{\Delta t} = \frac{f(3.01) - f(3)}{3.01 - 3} = \frac{16 \cdot 3.01^2 - 144}{0.01} = 96.16$
[3, 3.0001]	3	3.0001	0.0001	$\frac{\Delta s}{\Delta t} = \frac{f(3.0001) - f(3)}{3.0001 - 3} = \frac{16 \cdot 3.0001^2 - 144}{0.0001} = 96.0016$

The average velocity of 96.0016 over the time interval $\Delta t = 0.0001$ s should be very close to the instantaneous velocity of the rock at $t = 3$ s. As Δt gets closer to 0, the average velocity gets closer to the instantaneous velocity. So, to obtain the instantaneous velocity at $t = 3$ precisely, we use the limit of the average velocity as Δt approaches 0 or, equivalently, as t approaches 3.

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 3} \frac{f(t) - f(3)}{t - 3} = \lim_{t \to 3} \frac{16t^2 - 16 \cdot 3^2}{t - 3} = \lim_{t \to 3} \frac{16(t^2 - 9)}{t - 3}$$

$$= \lim_{t \to 3} \frac{16(t-3)(t+3)}{t - 3} = \lim_{t \to 3} [16(t+3)] = 96$$

The rock's instantaneous velocity at $t = 3$ s is 96 ft/s.

We generalize this result to obtain a definition for instantaneous velocity.

AP® CALC SKILL BUILDER FOR EXAMPLE 4

Finding Average Velocity

If the position of an object on the x-axis at the time t is $s(t) = \frac{1}{2}at^2 + t$ for a constant value a, find the average velocity of the object over the interval $0 \le t \le 5$.

Solution

Since $s(5) = \frac{1}{2}a(5)^2 + 5 = \frac{25}{2}a + 5$ and $s(0) = \frac{1}{2}a(0)^2 + 0 = 0$, the average velocity on the interval [0, 5] is

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(5) - s(0)}{5 - 0} = \frac{\frac{25}{2}a + 5 - 0}{5} = \frac{5}{2}a + 1$$

MATHEMATICAL PRACTICES TIP

Practice 4: Communication and Notation

Part of having proper notation is being able to identify the correct units that should be attached to a value. Write the formula for the average velocity $\left(\frac{\Delta s}{\Delta t}\right)$, and assume the units for s are meters and the units for t are hours. Ask students what the units are for Δs and Δt. What are the units for $\frac{\Delta s}{\Delta t}$? Recall that $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = v(t)$. Ask students what the units are for $v(t)$. Have them explain why the units for these values are both the same.

AP® CALC SKILL BUILDER FOR EXAMPLE 5

Finding Velocity

John throws a baseball straight up into the air. The ball's height above the ground at any time t is given by $h(t) = -16t^2 + 48t + 5$, where h is measured in feet and t is measured in seconds.

(a) Find the velocity of the ball at $t = 1$ s.

(b) Using correct units, interpret the meaning of your answer in the context of the problem.

Solution

(a) Use the definition of instantaneous velocity with $h(t) = -16t^2 + 48t + 5$ at $t = 1$.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 1} \frac{h(t) - h(1)}{t - 1}$$
$$= \lim_{t \to 1} \frac{-16t^2 + 48t + 5 - 37}{t - 1}$$
$$= \lim_{t \to 1} \frac{-16t^2 + 48t - 32}{t - 1}$$
$$= \lim_{t \to 1} \frac{-16\left(t^2 - 3t + 2\right)}{t - 1}$$
$$= \lim_{t \to 1} \frac{-16(t-1)(t-2)}{t - 1}$$
$$= \lim_{t \to 1} [-16(t-2)]$$
$$= 16$$

(b) At $t = 1$ s, the ball is traveling up with a velocity of 16 ft/s.

DEFINITION Instantaneous Velocity

If $s = f(t)$ is the position function of an object at time t, the **instantaneous velocity** v of the object at time t_0 is defined as the limit of the average velocity $\frac{\Delta s}{\Delta t}$ as Δt approaches 0. That is,

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

provided the limit exists.

IN WORDS
- Average velocity is measured over an interval of time.
- Instantaneous velocity is measured at a particular instant of time.

We usually shorten "instantaneous velocity" and just use the word "velocity."

NOW WORK Problem 31.

EXAMPLE 5 Finding Velocity

Find the velocity v of the falling rock from Example 4 at:

(a) $t_0 = 1$ s after it begins to fall
(b) $t_0 = 7.4$ s, just before it hits the Colorado River
(c) At any time t_0.

Solution

(a) Use the definition of instantaneous velocity with $f(t) = 16t^2$ and $t_0 = 1$.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1} = \lim_{t \to 1} \frac{16t^2 - 16}{t - 1} = \lim_{t \to 1} \frac{16(t^2 - 1)}{t - 1}$$
$$= \lim_{t \to 1} \frac{16(t-1)(t+1)}{t - 1} = \lim_{t \to 1} [16(t+1)] = 32$$

At 1 s, the velocity of the rock is 32 ft/s.

(b) For $t_0 = 7.4$ s,

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 7.4} \frac{f(t) - f(7.4)}{t - 7.4} = \lim_{t \to 7.4} \frac{16t^2 - 16 \cdot (7.4)^2}{t - 7.4}$$
$$= \lim_{t \to 7.4} \frac{16[t^2 - (7.4)^2]}{t - 7.4} = \lim_{t \to 7.4} \frac{16(t - 7.4)(t + 7.4)}{t - 7.4}$$
$$= \lim_{t \to 7.4} [16(t + 7.4)] = 16(14.8) = 236.8$$

At 7.4 s, the velocity of the rock is 236.8 ft/s.

NOTE Did you know? 236.8 ft/s is more than 161 mi/h!

(c)
$$v = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{16t^2 - 16t_0^2}{t - t_0} = \lim_{t \to t_0} \frac{16\,(t - t_0)\,(t + t_0)}{t - t_0}$$
$$= 16 \lim_{t \to t_0} (t + t_0) = 32t_0$$

At t_0 seconds, the velocity of the rock is $32t_0$ ft/s. ■

NOW WORK Problem 33.

4 Find the Derivative of a Function at a Number

Slope of a tangent line, rate of change of a function, and velocity are all found using the same limit,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The common underlying idea is the mathematical concept of *derivative*.

TRM Section 2.1: Worksheet 1

This two-page worksheet contains 3 problems that help students see how to move from an average to an instantaneous rate of change.

DEFINITION Derivative of a Function at a Number

If $y = f(x)$ is a function and c is in the domain of f, then the **derivative** of f at c, denoted by $f'(c)$, is the number

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.

EXAMPLE 6 Finding the Derivative of a Function at a Number

Find the derivative of $f(x) = 2x^2 - 3x - 2$ at $x = 2$. That is, find $f'(2)$.

Solution

Using the definition of the derivative, we have

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(2x^2 - 3x - 2) - 0}{x - 2} \qquad f(2) = 2 \cdot 4 - 3 \cdot 2 - 2 = 0$$

$$= \lim_{x \to 2} \frac{(x - 2)(2x + 1)}{x - 2}$$

$$= \lim_{x \to 2} (2x + 1) = 5$$

■

NOW WORK Problem 23 and AP® Practice Problems 2 and 6.

So far we have given three interpretations of the derivative:

- *Geometric interpretation:* If $y = f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$.
- *Rate of change of a function interpretation:* If $y = f(x)$, the derivative $f'(c)$ is the rate of change of f at c.
- *Physical interpretation:* If the signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s = f(t)$, the derivative $f'(t_0)$ is the velocity of the object at time t_0.

EXAMPLE 7 Finding an Equation of a Tangent Line

(a) Find the derivative of $f(x) = \sqrt{2x}$ at $x = 8$.

(b) Use the derivative $f'(8)$ to find an equation of the tangent line to the graph of f at the point $(8, 4)$.

Solution

(a) The derivative of f at 8 is

$$f'(8) = \lim_{x \to 8} \frac{f(x) - f(8)}{x - 8} = \lim_{x \to 8} \frac{\sqrt{2x} - 4}{x - 8} = \lim_{x \to 8} \frac{(\sqrt{2x} - 4)(\sqrt{2x} + 4)}{(x - 8)(\sqrt{2x} + 4)}$$

$f(8) = \sqrt{2 \cdot 8} = 4$ Rationalize the numerator.

$$= \lim_{x \to 8} \frac{2x - 16}{(x - 8)(\sqrt{2x} + 4)} = \lim_{x \to 8} \frac{2(x - 8)}{(x - 8)(\sqrt{2x} + 4)} = \lim_{x \to 8} \frac{2}{\sqrt{2x} + 4} = \frac{1}{4}$$

(b) The slope of the tangent line to the graph of f at the point $(8, 4)$ is $f'(8) = \frac{1}{4}$. Using the point-slope form of a line, we get

$$y - 4 = f'(8)(x - 8) \qquad y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{4}(x - 8) \qquad f'(8) = \frac{1}{4}$$

$$y = \frac{1}{4}x + 2$$

■

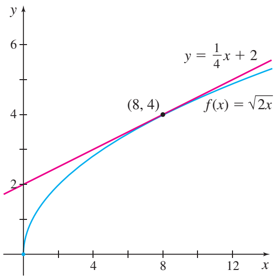


Figure 6

The graphs of f and the tangent line to the graph of f at $(8, 4)$ are shown in Figure 6.

NOW WORK Problem 15 and AP® Practice Problem 4.

Building Calculator Skills

You may want to introduce the students to the numerical derivative on the calculator. The function nDeriv will allow the students to use technology to find the value of the derivative of any function at any given point. This can help in checking their solutions in this section. It also builds up skill early, which can be helpful for the calculator portion of the AP® Calculus Exam.

TRM Section 2.1: Worksheet 2

This worksheet includes 2 problems that ask the student to find an equation for the tangent line to the graph of each function at a given point and 2 problems that ask the student to find the derivative of each function at a given number.

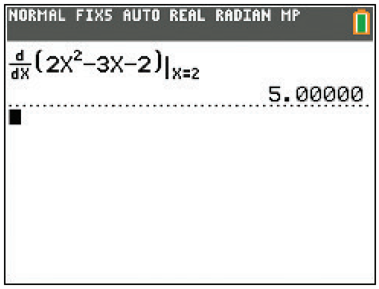
ALTERNATE EXAMPLE

Finding the Derivative of a Function at a Number

Find the value of the derivative of $f(x) = 2x^2 - 3x - 2$ at $x = 2$ using technology.

Solution

We use the nDeriv command found in the math menu.



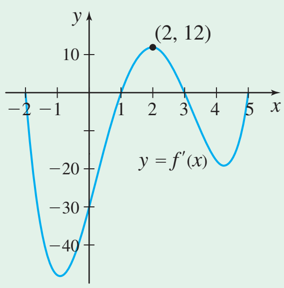
Note that the inputs are nDeriv(function, x, value to evaluate).

This result confirms the result in Example 6 obtained by calculating $f'(2)$ using the definition of the derivative.

AP® CALC SKILL BUILDER FOR EXAMPLE 7

Finding an Equation of a Tangent Line

The function f is defined on the closed interval $[-2, 5]$. The graph of the derivative $y = f'(x)$ is shown. Suppose the point $(2, 6)$ is on the graph of f. Find an equation of the tangent line to the graph of f at $(2, 6)$.



Solution

Since $f'(2) = 12$, the slope of the tangent line to the graph of f at $x = 2$ is 12. Using the point-slope form of the line, we get

$$y - f(2) = f'(2)(x - 2)$$

$$y - 6 = 12(x - 2)$$

$$y = 12x - 18$$

AP® CALC SKILL BUILDER FOR EXAMPLE 8

Approximating the Derivative of a Function Defined by a Table

A pot is filled with water and is then placed on a hot stove. The temperature of the water $W(t)$, in degrees Fahrenheit, is a continuous function of time t, in minutes. Values of $W(t)$ at selected times are given in the table.

t	0	2	4	6
$W(t)$	57	88	130	168

Use the data in the table to estimate $W'(2)$. Interpret the meaning of your answer in the context of the problem. Be sure to include units.

Solution

There are several ways to use these data to approximate the derivative at 2. We can use a secant line through the point corresponding to $t = 2$.

Using the secant line from 2 to 4, we get

$$W'(2) \approx m_{sec} = \frac{W(4)-W(2)}{4-2} = \frac{130-88}{4-2} = 21$$

On the AP® exam, students are expected to use a secant line through points whose values bracket the point $x = 2$, that is, the points through $x = 0$ and $x = 4$. Using the secant line from 0 to 4, we get:

$$W'(2) \approx m_{sec} = \frac{W(4)-W(0)}{4-0} = \frac{130-57}{4}$$
$$= 18.25$$

Using this estimate, the rate of increase of water temperature at $t = 2$ is approximately 18.25°F/min.

AP® EXAM TIP

Derivative problems tend to be presented to students on the exam in one of three ways. The student may be given a function, a graph, or a table and asked to calculate or estimate a derivative.

MUST-DO PROBLEMS FOR EXAM READINESS

AB: 7, 11, 15, 17, 23, 31, 33, 43, 51, and all AP® Practice Problems

BC: 11, 15, 17, 19, 23, 31, 33, 39, 43, 47, 51, 55, and all AP® Practice Problems

EXAMPLE 8 Approximating the Derivative of a Function Defined by a Table

The table below lists several values of a function $y = f(x)$ that is continuous on the interval $[-1, 5]$ and has a derivative at each number in the interval $(-1, 5)$. Approximate the derivative of f at 2.

x	0	1	2	3	4
$f(x)$	0	3	12	33	72

Solution

There are several ways to approximate the derivative of a function defined by a table. Each uses an average rate of change to approximate the rate of change of f at 2, which is the derivative of f at 2.

- Using the average rate of change from 2 to 3, we have
$$\frac{f(3)-f(2)}{3-2} = \frac{33-12}{1} = 21$$
With this choice, $f'(2)$ is approximately 21.
- Using the average rate of change from 1 to 2, we have
$$\frac{f(2)-f(1)}{2-1} = \frac{12-3}{1} = 9$$
With this choice, $f'(2)$ is approximately 9.
- A third approximation can be found by averaging the above two approximations.
Then $f'(2)$ is approximately $\frac{21+9}{2} = 15$. ■

NOW WORK Problem 51 and AP® Practice Problem 8.

2.1 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative is used to find instantaneous velocity.
2. *True or False* The derivative can be used to find the rate of change of a function.
3. The notation $f'(c)$ is read f ____ of c; $f'(c)$ represents the ____ of the tangent line to the graph of f at the point ____.
4. *True or False* If it exists, $\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$ is the derivative of the function f at 3.
5. If $f(x) = 6x - 3$, then $f'(3) =$ ________.
6. The velocity of an object, the slope of a tangent line, and the rate of change of a function are three different interpretations of the mathematical concept called the ________.

Skill Building

In Problems 7–16,

(a) *Find an equation for the tangent line to the graph of each function at the indicated point.*

(b) *Find an equation of the normal line to each function at the indicated point.*

(c) *Graph the function, the tangent line, and the normal line at the indicated point on the same set of coordinate axes.*

7. $f(x) = 3x^2$ at $(-2, 12)$
8. $f(x) = x^2 + 2$ at $(-1, 3)$
9. $f(x) = x^3$ at $(-2, -8)$
10. $f(x) = x^3 + 1$ at $(1, 2)$
11. [PAGE 163] $f(x) = \frac{1}{x}$ at $(1, 1)$
12. $f(x) = \sqrt{x}$ at $(4, 2)$
13. $f(x) = \frac{1}{x+5}$ at $\left(1, \frac{1}{6}\right)$
14. $f(x) = \frac{2}{x+4}$ at $\left(1, \frac{2}{5}\right)$
15. [PAGE 167] $f(x) = \frac{1}{\sqrt{x}}$ at $(1, 1)$
16. $f(x) = \frac{1}{x^2}$ at $(1, 1)$

In Problems 17–20, find the rate of change of f at the indicated numbers.

17. [PAGE 164] $f(x) = 5x - 2$ at (a) $c = 0$, (b) $c = 2$
18. $f(x) = x^2 - 1$ at (a) $c = -1$, (b) $c = 1$
19. $f(x) = \frac{x^2}{x+3}$ at (a) $c = 0$, (b) $c = 1$
20. $f(x) = \frac{x}{x^2-1}$ at (a) $c = 0$, (b) $c = 2$

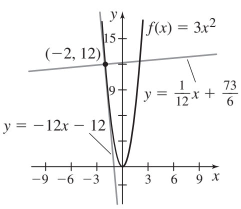
In Problems 21–30, find the derivative of each function at the given number.

21. $f(x) = 2x + 3$ at 1
22. $f(x) = 3x - 5$ at 2
23. [PAGE 167] $f(x) = x^2 - 2$ at 0
24. $f(x) = 2x^2 + 4$ at 1
25. $f(x) = 3x^2 + x + 5$ at -1
26. $f(x) = 2x^2 - x - 7$ at -1
27. $f(x) = \sqrt{x}$ at 4
28. $f(x) = \frac{1}{x^2}$ at 2
29. $f(x) = \frac{2-5x}{1+x}$ at 0
30. $f(x) = \frac{2+3x}{2+x}$ at 1

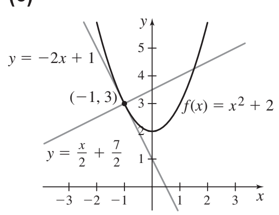
TRM Full Solutions to Section 2.1 Problems and AP® Practice Problems

Answers to Section 2.1 Problems

1. True.
2. True.
3. Prime, slope, $(c, f(c))$
4. True.
5. 6
6. Derivative.
7. (a) $y = -12x - 12$
(b) $y = \frac{1}{12}x + \frac{73}{6}$
(c)



8. (a) $y = -2x + 1$
(b) $y = \frac{x}{2} + \frac{7}{2}$
(c)



Answers continue on p. 169

PAGE 166 **31. Approximating Velocity** An object in rectilinear motion moves according to the position function $s(t) = 10t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 3$ s by letting Δt first equal 0.1 s, then 0.01 s, and finally 0.001 s. What limit does the velocity appear to be approaching? Organize the results in a table.

32. Approximating Velocity An object in rectilinear motion moves according to the position function $s(t) = 5 - t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 1$ by letting Δt first equal 0.1, then 0.01, and finally 0.001. What limit does the velocity appear to be approaching? Organize the results in a table.

PAGE 166 **33. Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 3t^2 + 4t$. Find the velocity v at $t_0 = 0$. At $t_0 = 2$. At any time t_0.

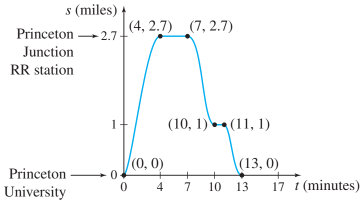
34. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 2t^3 + 4$. Find the velocity v at $t_0 = 0$. At $t_0 = 3$. At any time t_0.

35. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 3t^2 - \frac{1}{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

36. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 2\sqrt{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

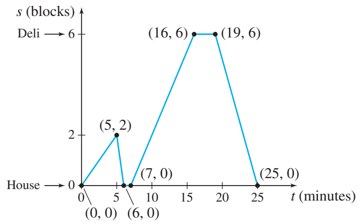
37. The Princeton Dinky is the shortest rail line in the country. It runs for 2.7 miles, connecting Princeton University to the Princeton Junction railroad station. The Dinky starts from the university and moves north toward Princeton Junction. Its distance from Princeton is shown in the graph (top, right), where the time t is in minutes and the distance s of the Dinky from Princeton University is in miles.

(a) When is the Dinky headed toward Princeton University?
(b) When is it headed toward Princeton Junction?
(c) When is the Dinky stopped?
(d) Find its average velocity on a trip from Princeton to Princeton Junction.
(e) Find its average velocity for the round-trip shown in the graph, that is, from $t = 0$ to $t = 13$.



38. Barbara walks to the deli, which is six blocks east of her house. After walking two blocks, she realizes she left her phone on her desk, so she runs home. After getting the phone, and closing and locking the door, Barbara starts on her way again. At the deli, she waits in line to buy a bottle of **vitaminwater**™, and then she jogs home. The graph below represents Barbara's journey. The time t is in minutes, and s is Barbara's distance, in blocks, from home.

(a) At what times is she headed toward the deli?
(b) At what times is she headed home?
(c) When is the graph horizontal? What does this indicate?
(d) Find Barbara's average velocity from home until she starts back to get her phone.
(e) Find Barbara's average velocity from home to the deli after getting her phone.
(f) Find her average velocity from the deli to home.



Applications and Extensions

39. Slope of a Tangent Line An equation of the tangent line to the graph of a function f at (2, 6) is $y = -3x + 12$. What is $f'(2)$?

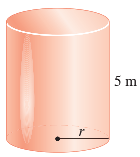
40. Slope of a Tangent Line An equation of the tangent line of a function f at (3, 2) is $y = \frac{1}{3}x + 1$. What is $f'(3)$?

41. Tangent Line Does the tangent line to the graph of $y = x^2$ at (1, 1) pass through the point (2, 5)?

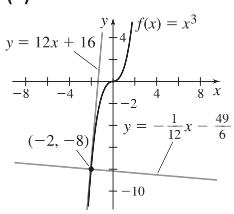
42. Tangent Line Does the tangent line to the graph of $y = x^3$ at (1, 1) pass through the point (2, 5)?

PAGE 164 **43. Respiration Rate** A human being's respiration rate R (in breaths per minute) is given by $R = R(p) = 10.35 + 0.59p$, where p is the partial pressure of carbon dioxide in the lungs. Find the rate of change in respiration when $p = 50$.

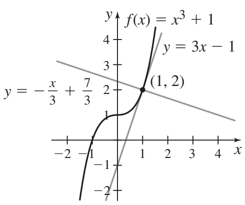
44. Instantaneous Rate of Change The volume V of the right circular cylinder of height 5 m and radius r m shown in the figure is $V = V(r) = 5\pi r^2$. Find the instantaneous rate of change of the volume with respect to the radius when $r = 3$ m.



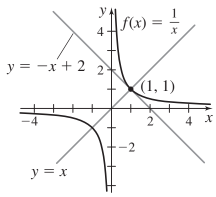
9. (a) $y = 12x + 16$
(b) $y = -\frac{1}{12}x - \frac{49}{6}$
(c)



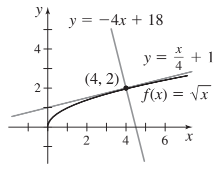
10. (a) $y = 3x - 1$
(b) $y = -\frac{x}{3} + \frac{7}{3}$
(c)



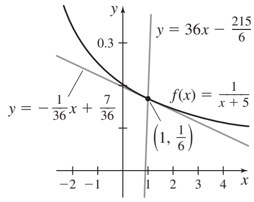
11. (a) $y = -x + 2$
(b) $y = x$
(c)



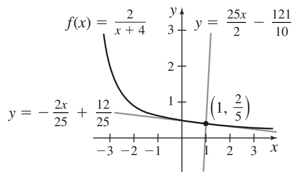
12. (a) $y = \frac{x}{4} + 1$
(b) $y = -4x + 18$
(c)



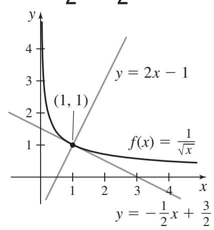
13. (a) $y = -\frac{1}{36}x + \frac{7}{36}$
(b) $y = 36x - \frac{215}{6}$
(c)



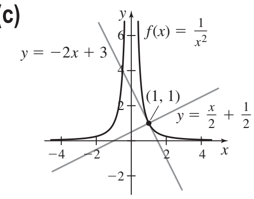
14. (a) $y = -\frac{2x}{25} + \frac{12}{25}$
(b) $y = \frac{25x}{2} - \frac{121}{10}$
(c)



15. (a) $y = -\frac{1}{2}x + \frac{3}{2}$ **(b)** $y = 2x - 1$
(c)



16. (a) $y = -2x + 3$
(b) $y = \frac{x}{2} + \frac{1}{2}$
(c)



17. (a) 5 **(b)** 5
18. (a) −2 **(b)** 2
19. (a) 0 **(b)** $\frac{7}{16}$
20. (a) −1 **(b)** $-\frac{5}{9}$

Answers continue on p. 170

45. Market Share During a month-long advertising campaign, the total sales S of a magazine is modeled by the function $S(x) = 5x^2 + 100x + 10{,}000$, where x, $0 \le x \le 30$, represents the number of days since the campaign began.

(a) What is the average rate of change of sales from $x = 10$ to $x = 20$ days?

(b) What is the instantaneous rate of change of sales when $x = 10$ days?

46. Demand Equation The demand equation for an item is $p = p(x) = 90 - 0.02x$, where p is the price in dollars and x is the number of units (in thousands) made.

(a) Assuming all units made can be sold, find the revenue function $R(x) = xp(x)$.

(b) Marginal Revenue Marginal revenue is defined as the additional revenue earned by selling an additional unit. If we use $R'(x)$ to measure the marginal revenue, find the marginal revenue when 1 million units are sold.

47. Gravity If a ball is dropped from the top of the Empire State Building, 1002 ft above the ground, the distance s (in feet) it falls after t seconds is $s(t) = 16t^2$.

(a) What is the average velocity of the ball for the first 2 s?

(b) How long does it take for the ball to hit the ground?

(c) What is the average velocity of the ball during the time it is falling?

(d) What is the velocity of the ball when it hits the ground?

48. Velocity A ball is thrown upward. Its height h in feet is given by $h(t) = 100t - 16t^2$, where t is the time elapsed in seconds.

(a) What is the velocity v of the ball at $t = 0$ s, $t = 1$ s, and $t = 4$ s?

(b) At what time t does the ball strike the ground?

(c) At what time t does the ball reach its highest point? *Hint:* At the time the ball reaches its maximum height, it is stationary. So, its velocity $v = 0$.

49. Gravity A rock is dropped from a height of 88.2 m and falls toward Earth in a straight line. In t seconds the rock falls $4.9t^2$ m.

(a) What is the average velocity of the rock for the first 2 s?

(b) How long does it take for the rock to hit the ground?

(c) What is the average velocity of the rock during its fall?

(d) What is the velocity v of the rock when it hits the ground?

50. Velocity At a certain instant, the speedometer of an automobile reads V mi/h. During the next $\frac{1}{4}$ s the automobile travels 20 ft. Approximate V from this information.

PAGE 168 **51.** A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 h the water is continuously used. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

(a) Use the table to approximate $A'(5)$.

(b) Using appropriate units, interpret $A'(5)$ in the context of the problem.

52. The table below lists the outside temperature T, in degrees Fahrenheit, in Naples, Florida, on a certain day in January, for selected times x, where x is the number of hours since 12 a.m.

x	5	7	9	11	12	13	14	16	17
$T(x)$	62	71	74	78	81	83	84	85	78

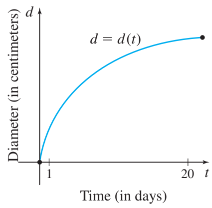
(a) Use the table to approximate $T'(11)$.

(b) Using appropriate units, interpret $T'(11)$ in the context of the problem.

53. Rate of Change Show that the rate of change of a linear function $f(x) = mx + b$ is the slope m of the line $y = mx + b$.

54. Rate of Change Show that the rate of change of a quadratic function $f(x) = ax^2 + bx + c$ is a linear function of x.

55. Agriculture The graph represents the diameter d (in centimeters) of a maturing peach as a function of the time t (in days) it is on the tree.

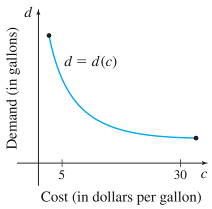


(a) Interpret the derivative $d'(t)$ as a rate of change.

(b) Which is larger, $d'(1)$ or $d'(20)$?

(c) Interpret both $d'(1)$ and $d'(20)$.

56. Business The graph represents the demand d (in gallons) for olive oil as a function of the cost c (in dollars per gallon) of the oil.



(a) Interpret the derivative $d'(c)$.

(b) Which is larger, $d'(5)$ or $d'(30)$? Give an interpretation to $d'(5)$ and $d'(30)$.

57. Volume of a Cube A metal cube with each edge of length x centimeters is expanding uniformly as a consequence of being heated.

(a) Find the average rate of change of the volume of the cube with respect to an edge as x increases from 2.00 to 2.01 cm.

(b) Find the instantaneous rate of change of the volume of the cube with respect to an edge at the instant when $x = 2$ cm.

21. $f'(1) = 2$ **22.** $f'(0) = 3$

23. $f'(0) = 0$ **24.** $f'(1) = 4$

25. $f'(-1) = -5$ **26.** $f'(-1) = -5$

27. $f'(4) = \frac{1}{4}$ **28.** $f'(2) = -\frac{1}{4}$

29. $f'(0) = -7$ **30.** $f'(1) = \frac{4}{9}$

31.

Time interval	Δt	$\frac{\Delta s}{\Delta t}$
$[3, 3.1]$	0.1	61
$[3, 3.01]$	0.01	60.1
$[3, 3.001]$	0.001	60.01

Velocity appears to approach 60 cm/s.

32.

Time interval	Δt	$\frac{\Delta s}{\Delta t}$
$[1, 1.1]$	0.1	−2.1
$[1, 1.01]$	0.01	−2.01
$[1, 1.001]$	0.001	−2.001

Velocity appears to approach −2 cm/s.

33. $v(0) = 4$ m/s, $v(2) = 16$ m/s, $v(t_0) = 6t_0 + 4$ m/s

34. $v(0) = 0$ m/s, $v(3) = 54$ m/s, $v(t_0) = 6(t_0)^2$ m/s

35. $v(1) = 7$ cm/s, $v(4) = \frac{385}{16}$ cm/s

36. $v(1) = 1$ cm/s, $v(4) = \frac{1}{2}$ cm/s

37. (a) $7 \le t \le 10$ and $11 \le t \le 13$
(b) $0 \le t \le 4$
(c) $4 \le t \le 7$ and $10 \le t \le 11$
(d) 0.675 mi/min **(e)** 0.415 mi/min

38. (a) $0 \le t \le 5$ and $7 \le t \le 16$
(b) $5 \le t \le 6$ and $19 \le t \le 25$
(c) $6 \le t \le 7$ and $16 \le t \le 19$. Barbara is stationary.
(d) $\frac{2}{5}$ block/min
(e) $\frac{2}{3}$ block/min **(f)** −1 block/min

39. −3 **40.** $\frac{1}{3}$

41. No. **42.** No.

43. $R'(50) = 0.59$ **44.** $V'(3) = 30\pi$ m^3/m

45. (a) 250 sales per day.
(b) 200 sales per day.

46. (a) $R(x) = 90x - 0.02x^2$
(b) \$50/(thousand units)

47. (a) 32 ft/s **(b)** 7.914 s
(c) ≈ 126.618 ft/s **(d)** ≈ 253.235 ft/s

48. (a) 100 ft/s, 68 ft/s, −28 ft/s
(b) 6.25 s **(c)** 3.125 s

49. (a) 9.8 m/s **(b)** ≈ 4.243 s
(c) ≈ 20.789 m/s **(d)** ≈ 41.578 m/s

50. ~54.5 mi/h

51. (a) $A'(t) = \frac{66-71}{5-2} = \frac{-5}{3}$ using (2,71) to (5,66)

$A'(t) = \frac{60-66}{7-5} = \frac{6}{-2} = -3$ using (5,66) to (7,60)

(b) $A'(t)$ is approximately between $\frac{-5}{3}$ and -3 inclusive.

$A'(t)$ is the rate at which the volume of the water in liters is decreasing over the time in hours.

The volume of water in the tank is decreasing at a rate between $\frac{5}{3}$ and 3 inclusive liters per hour at 12:00 noon.

Answers continue on p. 171

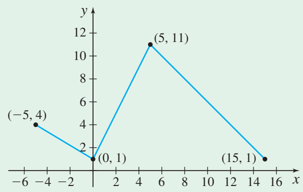
Preparing for the AP® Exam

AP® Practice Problems

PAGE 163 **1.** The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. The values $f(2)$ and $f'(2)$ are:

(A) $f(2) = 2;\ f'(2) = -1$ (B) $f(2) = 3;\ f'(2) = -1$
(C) $f(2) = 2;\ f'(2) = 1$ (D) $f(2) = 3;\ f'(2) = 2$

PAGE 167 **2.** The graph of the function f, given below, consists of three line segments. Find $f'(3)$.



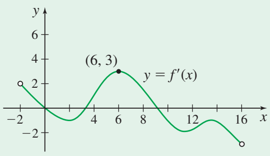
Graph of f

(A) 1 (B) 2 (C) 3 (D) $f'(3)$ does not exist

PAGE 164 **3.** What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$?

(A) 5 (B) 7 (C) 12 (D) 17

PAGE 167 **4.** The function f is defined on the closed interval $[-2, 16]$. The graph of the derivative of f, $y = f'(x)$, is given below.



The point $(6, -2)$ is on the graph of $y = f(x)$. An equation of the tangent line to the graph of f at $(6, -2)$ is

(A) $y = 3$ (B) $y + 2 = 6(x + 3)$
(C) $y + 2 = 6x$ (D) $y + 2 = 3(x - 6)$

PAGE 163 **5.** If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point $(2, 6)$, then $f'(2) =$

(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{13}{3}$

PAGE 167 **6.** If f is a function for which $\lim_{x \to -3} \frac{f(x) - f(-3)}{x + 3} = 0$, then which of the following statements must be true?

(A) $x = -3$ is a vertical asymptote of the graph.
(B) The derivative of f at $x = -3$ exists.
(C) The function f is continuous at $x = 3$.
(D) f is not defined at $x = -3$.

PAGE 165 **7.** If the position of an object on the x-axis at time t is $4t^2$, then the average velocity of the object over the interval $0 \le t \le 5$ is

(A) 5 (B) 20 (C) 40 (D) 100

PAGE 168 **8.** A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

Use the table to approximate $A'(5)$.

2.2 The Derivative as a Function; Differentiability

OBJECTIVES *When you finish this section, you should be able to:*

1 **Define the derivative function (p. 171)**
2 **Graph the derivative function (p. 173)**
3 **Identify where a function is not differentiable (p. 175)**

1 Define the Derivative Function

The derivative of f at a real number c has been defined as the real number

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad (1)$$

provided the limit exists. We refer to this representation of the derivative as Form (1).

Answers to AP® Practice Problems

1. B
2. B
3. C
4. D
5. C
6. B
7. B
8. Answers will vary.

TRM Alternate Examples Section 2.2

You can find the Alternate Examples for this section in PDF format in the Teacher's Resource Materials.

TRM AP® Calc Skill Builders Section 2.2

You can find the AP® Calc Skill Builders for this section in PDF format in the Teacher's Resource Materials.

52. **(a)** $T'(11) = \frac{78-74}{11-9} = \frac{4}{2} = 2$ using (9, 74) and (11, 78)

$T'(11) = \frac{81-78}{12-11} = 3$ using (11, 78) and (12, 81)

(b) $A'(t)$ is approximately between 2 and 3 inclusive.
$A'(t)$ is the rate at which the temperature is increasing in degrees Fahrenheit per hour at 11:00 a.m. on a certain day in January in Naples, Florida.

53. See TSM for proof.

54. $f'(x) = 2ax + b$

55. **(a)** $d'(t)$ is rate of change of diameter (cm) with respect to time (days).

(b) $d'(1) > d'(20)$

(c) $d'(1)$ is instantaneous rate of change of peach's diameter on day 1 and $d'(20)$ is instantaneous rate of change of peach's diameter on day 20.

56. **(a)** $d'(c)$ is rate of change of demand as a function of cost per gallon of olive oil.

(b) $d'(30) > d'(5)$; $d'(30)$ looks slightly less than 0, so when the price is \$30/gal, the demand goes down a little for every \$1 the price goes up; looks a lot less than 0, so when the price is \$5/gal, the demand goes down a lot for every \$1 the price goes up.

57. **(a)** $\frac{\Delta V}{\Delta x} \circ 12.060\ \text{cm}^3/\text{cm}$

(b) $V'(2) = 12\ \text{cm}^3/\text{cm}$

Teaching Tip

It is possible to teach this section in conjunction with Section 2.1. If you are pressed for time, the concepts of instantaneous rate of change, tangent lines, and normal lines can be covered in Section 2.3 by addressing them once the students can take derivatives using the Power Rule.

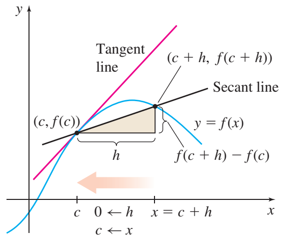


Figure 7 The slope of the tangent line at c is $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Another way to find the derivative of f at any real number is obtained by rewriting the expression $\frac{f(x) - f(c)}{x - c}$ and letting $x = c + h$, $h \neq 0$. Then

$$\frac{f(x) - f(c)}{x - c} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h}$$

Since $x = c + h$, then as x approaches c, h approaches 0. Form (1) of the derivative with these changes becomes

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

So now, we have an equivalent way to write Form (1) for the derivative of f at a real number c.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

See Figure 7.

In the expression above for $f'(c)$, c is any real number. That is, the derivative f' is a function, called the *derivative function* of f. Now replace c by x, the independent variable of f.

Teaching Tip

This is the definition of a derivative that students will use most often in the course. You can use this definition to derive derivative rules later in the class.

DEFINITION The Derivative Function f'

The **derivative function** f' of a function f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (2)$$

provided the limit exists. If f has a derivative, then f is said to be **differentiable**.

IN WORDS In Form (2) the derivative is the limit of a difference quotient.

We refer to this representation of the derivative as Form (2).

EXAMPLE 1 Finding the Derivative Function

Find the derivative of the function $f(x) = x^2 - 5x$ at any real number x using Form (2).

Solution

Using Form (2), we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 - 5(x+h)] - (x^2 - 5x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x^2 + 2xh + h^2) - 5x - 5h] - x^2 + 5x}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 5)}{h} = \lim_{h \to 0} (2x + h - 5) = 2x - 5$$

■

NOTE Compare the solution and answer found in Example 1 to Example 2(b) on pp. 163–164.

NOW WORK AP® Practice Problem 2.

The domain of the function f' is the set of real numbers in the domain of f for which the limit (2) exists. So the domain of f' is a subset of the domain of f.

We can use either Form (1) or Form (2) to find derivatives. However, if we want the derivative of f at a specified number c, we usually use Form (1) to find $f'(c)$. If we want to find the derivative function of f, we usually use Form (2) to find $f'(x)$. In this section, we use the definitions of the derivative, Forms (1) and (2), to investigate derivatives. In the next section, we begin to develop formulas for finding the derivatives.

SUGGESTED SKILL 1.D

Students should be able to identify the best procedure to use when they are presented with a problem. Students have now been presented with two forms for finding a derivative. Form 1 is used to find the derivative of a function at a number, while Form 2 is used to find the derivative function. Have a conversation with students about which form they should use in different situations so that they can begin to think about using different procedures for different problems.

AP® CALC SKILL BUILDER FOR EXAMPLE 1

Finding the Derivative of a Function at a Number c

Find the derivative of the function $f(x) = 2x^2 + 3x - 1$ at any real number x.

Using Form (2),

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0} \frac{[2(c+h)^2 + 3(c+h) - 1] - (2c^2 + 3c - 1)}{h}$$

$$= \lim_{h \to 0} \frac{2c^2 + 4ch + 2h^2 + 3c + 3h - 1 - 2c^2 - 3c + 1}{h}$$

$$= \lim_{h \to 0} \frac{4ch + 2h^2 + 3h}{h}$$

$$= \lim_{h \to 0} \frac{h(4c + 2h + 3)}{h} = \lim_{h \to 0} (4c + 2h + 3) = 4c + 3$$

Thus, $f'(c) = 4c + 3$ for any real number c.

EXAMPLE 2 **Finding the Derivative Function**

NOTE The instruction "differentiate f" means "find the derivative of f."

Differentiate $f(x)=\sqrt{x}$ and determine the domain of f'.

Solution

The domain of f is $\{x|x \geq 0\}$. To find the derivative of f, we use Form (2). Then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

We rationalize the numerator to find the limit.

$$f'(x)=\lim_{h\to 0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

The limit does not exist when $x=0$. But for all other x in the domain of f, the limit does exist. So, the domain of the derivative function $f'(x)=\dfrac{1}{2\sqrt{x}}$ is $\{x|x>0\}$. ■

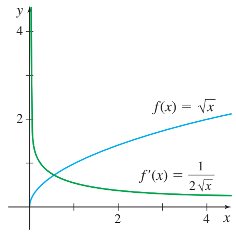


Figure 8

In Example 2, notice that the domain of the derivative function f' is a proper subset of the domain of the function f. The graphs of both f and f' are shown in Figure 8.

NOW WORK Problem 15.

EXAMPLE 3 **Interpreting the Derivative as a Rate of Change**

The surface area S (in square meters) of a balloon is expanding as a function of time t (in seconds) according to $S=S(t)=5t^2$. Find the rate of change of the surface area of the balloon with respect to time. What are the units of $S'(t)$?

Solution

An interpretation of the derivative function $S'(t)$ is the rate of change of $S=S(t)$.

$$S'(t)=\lim_{h\to 0}\frac{S(t+h)-S(t)}{h}=\lim_{h\to 0}\frac{5(t+h)^2-5t^2}{h} \quad \text{Form (2)}$$

$$=\lim_{h\to 0}\frac{5(t^2+2th+h^2)-5t^2}{h}$$

$$=\lim_{h\to 0}\frac{5t^2+10th+5h^2-5t^2}{h}=\lim_{h\to 0}\frac{10th+5h^2}{h}$$

$$=\lim_{h\to 0}\frac{(10t+5h)h}{h}=\lim_{h\to 0}(10t+5h)=10t$$

Since $S'(t)$ is the limit of the quotient of a change in area divided by a change in time, the units of the rate of change are square meters per second (m^2/s). The rate of change of the surface area S of the balloon with respect to time is $10t$ m^2/s. ■

NOW WORK Problem 67 and AP® Practice Problems 10 and 11.

2 Graph the Derivative Function

There is a relationship between the graph of a function and the graph of its derivative.

EXAMPLE 4 **Graphing a Function and Its Derivative**

Find f' if $f(x)=x^3-1$. Then graph $y=f(x)$ and $y=f'(x)$ on the same set of coordinate axes.

AP® CALC SKILL BUILDER FOR EXAMPLE 3

Interpreting the Derivative as a Rate of Change

The function F models the number of people in a football stadium, in thousands, where t is the number of hours after noon. Which of the following is the best interpretation of the statement $F'(3)=35$?

(a) At noon, the number of people in the football stadium was increasing at a rate of 35,000 people per hour.

(b) At 3:00 p.m., the number of people in the football stadium was increasing at a rate of 35,000 people per hour.

(c) At noon, the number of people in the football stadium was 35,000.

(d) At 3:00 p.m., the number of people in the football stadium was 35,000.

(e) From noon until 3:00 p.m., the number of people in the football stadium was increasing at an average rate of 35,000 people per hour.

Solution

$F'(3)$ is the rate of change of F at $t=3$. Since $t=3$ is the time 3 hours after noon, $F'(3)$ is the rate of change in the number of people in the stadium at 3 p.m. Since $F'(3)=35$, the number of people in the stadium is increasing at a rate of 35,000 people per hour at 3 p.m. The answer is b.

MATHEMATICAL PRACTICES TIP

Practice 4: Communication and Notation

You should have students frequently explain what their answers mean using mathematically precise language. Students should be able to explain what the derivative of the function means in the context of the problem. Many students have difficulty interpreting what their answers mean, so be sure to model good explanations for your students, as in the AP® Calc Skill Builder above.

AP® CALC SKILL BUILDER FOR EXAMPLE 2

Finding the Derivative of a Function

Differentiate $g(x)=\dfrac{1}{x}$ and determine the domain of g'.

Solution

The domain of g is $\{x \mid x\neq 0\}$. To find the derivative of g, we use Form (2). Then

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\frac{-h}{x(x+h)}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{-1}{x(x+h)}=\frac{-1}{x^2}$$

The limit exists for all x in the domain of g. Thus, the domain of the derivative function $g'(x)=-\dfrac{1}{x^2}$ is $\{x \mid x\neq 0\}$.

TRM Section 2.2: Worksheet 1

This worksheet contains 2 questions in which the students are asked to differentiate the given function, then graph the function as well as its derivative on the coordinate planes provided.

TRM Section 2.2: Worksheet 2

This worksheet contains 2 questions in which the students are asked to differentiate the given function, then graph the function as well as its derivative on the coordinate planes provided.

Teaching Tip

Try this activity to help students make a connection between the graph of a function and the graph of its derivative. Ask the students to create 10 coordinate axes in two rows of five using the same scale.

On the coordinate planes in row 1, have the students graph the following five equations:

$$y = 1,\ y = x,\ y = x^2,\ y = x^3,\ y = x^4$$

On the coordinate planes in row 2, have the students graph the following five equations:

$$y = 0,\ y = 1,\ y = 2x,\ y = 3x^2,\ y = 4x^3$$

Ask them about the slope of each of the graphs in the first row. The first two should make sense and help to lead to an understanding of the other three. Positioning the graphs above or below one another will align the values on the *x*-axis and allow for easy comparison between the graphs.

Solution

$f(x) = x^3 - 1$ so

$$f(x+h) = (x+h)^3 - 1 = x^3 + 3hx^2 + 3h^2x + h^3 - 1$$

Using Form (2), we find

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x^3 + 3hx^2 + 3h^2x + h^3 - 1) - (x^3 - 1)}{h}$$

$$= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3hx + h^2)}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3hx + h^2) = 3x^2$$

The graphs of f and f' are shown below in Figure 9. ■

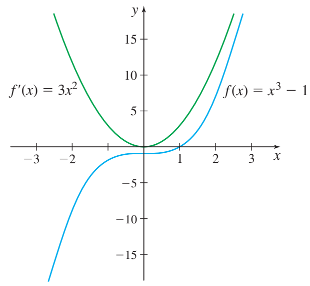


Figure 9

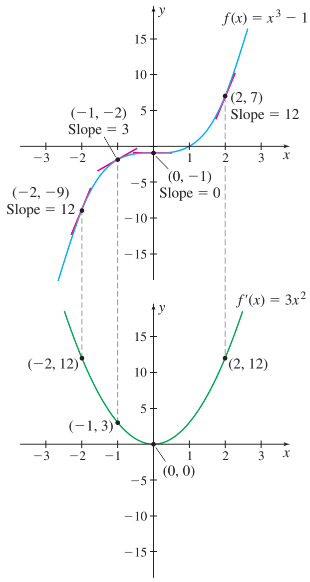


Figure 10

NOW WORK Problem 19.

Figure 10 illustrates several tangent lines to the graph of $f(x) = x^3 - 1$. Observe that the tangent line to the graph of f at $(0, -1)$ is horizontal, so its slope is 0. Then $f'(0) = 0$, so the graph of f' contains the point $(0, 0)$. Also notice that every tangent line to the graph of f has a nonnegative slope, so $f'(x) \geq 0$. That is, the range of the function f' is $\{y \mid y \geq 0\}$. Finally, notice that the slope of each tangent line is the y-coordinate of the corresponding point on the graph of the derivative f'.

With these ideas in mind, we can obtain a rough sketch of the derivative function f', even if we know only the graph of the function f.

EXAMPLE 5 Graphing the Derivative Function

Use the graph of the function $y = f(x)$, shown in Figure 11, to sketch the graph of the derivative function $y = f'(x)$.

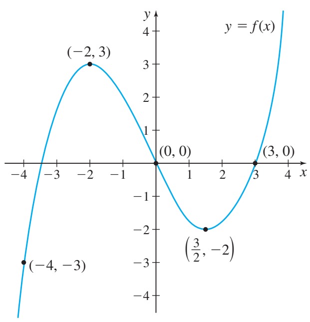


Figure 11

Solution

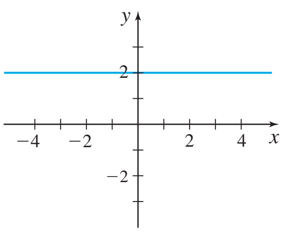
We begin by drawing tangent lines to the graph of f at the points shown in Figure 11. See the graph at the top of Figure 12. At the points $(-2, 3)$ and $\left(\frac{3}{2}, -2\right)$ the tangent lines are horizontal, so their slopes are 0. This means $f'(-2) = 0$ and $f'\left(\frac{3}{2}\right) = 0$, so the points $(-2, 0)$ and $\left(\frac{3}{2}, 0\right)$ are on the graph of the derivative function at the bottom of Figure 12. Now we estimate the slope of the tangent lines at the other selected points.

ALTERNATE EXAMPLE

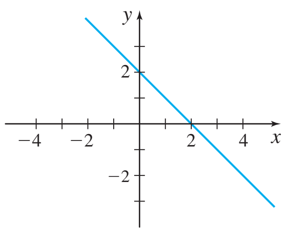
Graphing the Derivative Function

Suppose f is a quadratic function. Which of the following could be the graph of f'?

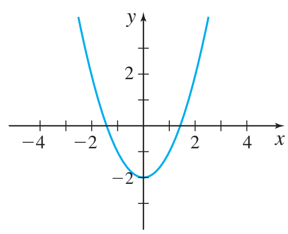
(a)



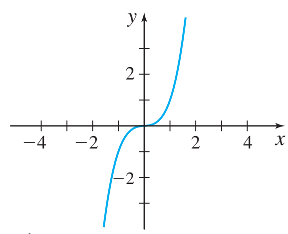
(b)



(c)



(d)



Solution

If f is a quadratic function, we know the function has the form $f(x) = ax^2 + bx + c$, for a, b, c constants. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a((x+h)^2 + b(x+h) + c) - (ax^2 + bx + c)}{h}$$

$$= \lim_{h\to 0} \frac{(ax^2 + 2ahx + ah^2 + bx + bh + c) - ax^2 - bx - c}{h}$$

$$= \lim_{h\to 0} \frac{(2ahx + ah^2 + bh)}{h} = \lim_{h\to 0} 2ax + ah + b = 2ax + b$$

Thus, $f'(x) = 2ax + b$, which is the equation of a line. The derivative of f must be a linear function. The answer is b.

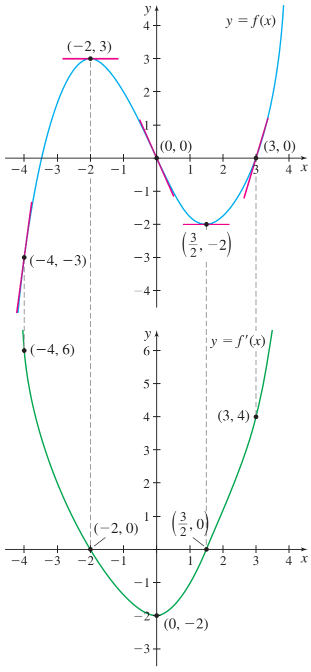


Figure 12

For example, at the point $(-4, -3)$, the slope of the tangent line is positive and the line is rather steep. We estimate the slope to be close to 6, and we plot the point $(-4, 6)$ on the bottom graph of Figure 12. Continue the process and then connect the points with a smooth curve. ■

Notice in Figure 12 that at the points on the graph of f where the tangent lines are horizontal, the graph of the derivative f' intersects the x-axis. Also notice that wherever the graph of f is increasing, the slopes of the tangent lines are positive, that is, f' is positive, so the graph of f' is above the x-axis. Similarly, wherever the graph of f is decreasing, the slopes of the tangent lines are negative, so the graph of f' is below the x-axis.

NOW WORK Problem 29.

3 Identify Where a Function Is Not Differentiable

Suppose a function f is continuous on an open interval containing the number c. The function f is not differentiable at the number c if $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ does not exist. Three (of several) ways this can happen are:

- $\lim\limits_{x\to c^-}\dfrac{f(x)-f(c)}{x-c}$ exists and $\lim\limits_{x\to c^+}\dfrac{f(x)-f(c)}{x-c}$ exists, but they are not equal.*
When this happens the graph of f has a **corner** at $(c, f(c))$. For example, the absolute value function $f(x)=|x|$ has a corner at $(0, 0)$. See Figure 13.
- The one-sided limits are both infinite and both equal ∞ or both equal $-\infty$. When this happens, the graph of f has a vertical tangent line at $(c, f(c))$. For example, the cube root function $f(x)=\sqrt[3]{x}$ has a vertical tangent at $(0, 0)$. See Figure 14.
- Both one-sided limits are infinite, but one equals $-\infty$ and the other equals ∞. When this happens, the graph of f has a vertical tangent line at the point $(c, f(c))$. This point is referred to as a **cusp**. For example, the function $f(x)=x^{2/3}$ has a cusp at $(0, 0)$. See Figure 15.

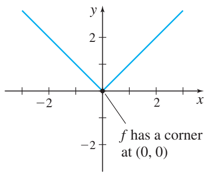


Figure 13 $f(x)=|x|$; $f'(0)$ does not exist.

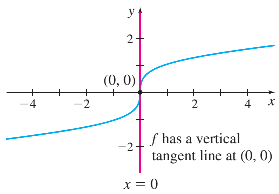


Figure 14 $f(x)=\sqrt[3]{x}$; $f'(0)$ does not exist.

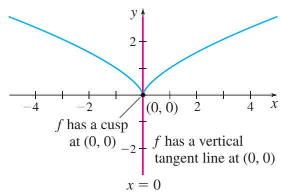


Figure 15 $f(x)=x^{2/3}$; $f'(0)$ does not exist.

EXAMPLE 6 Identifying Where a Function Is Not Differentiable

Given the piecewise defined function $f(x)=\begin{cases}-2x^2+4 & \text{if } x<1\\ x^2+1 & \text{if } x\geq 1\end{cases}$, determine whether $f'(1)$ exists.

Solution

Use Form (1) of the definition of a derivative to determine whether $f'(1)$ exists.

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{f(x)-2}{x-1} \qquad f(1)=1^2+1=2$$

*The one-sided limits, $\lim\limits_{x\to c^-}\dfrac{f(x)-f(c)}{x-c}$ and $\lim\limits_{x\to c^+}\dfrac{f(x)-f(c)}{x-c}$, are called the **left-hand derivative** of f at c, denoted $f'_-(c)$, and the **right-hand derivative** of f at c, denoted $f'_+(c)$, respectively. Using properties of limits, the derivative $f'(c)$ exists if and only if $f'_-(c)=f'_+(c)$.

Teaching Tip

Have students explore points where a function is not differentiable using this activity. Ask students to draw a coordinate plane and plot the points (–4, 0), (–2, 0), (2, 0), and (4, 0). Direct the students to connect the points with a continuous function with lines or curves so that at three of the four points the derivative does not exist. As an extra challenge, ask them to use each of the three ways that a derivative cannot exist: corner, cusp, and vertical tangent. If they can do that, ask them to try to do it with three points. Discuss.

Teaching Tip

Have students explore the derivative of piecewise functions with this activity. Have the students write down two numbers between 1 and 5. They may write the same number twice. Then reveal this list of functions:

1. Absolute value
2. Sin(x) or cos(x)
3. Parabola
4. Semicircle
5. Constant or linear

The challenge is to use any variety of the two functions they randomly selected for themselves and see if they can create a piecewise function that is differentiable on its domain. Ask them to share their stories of success (or lack of success) with the others in the class.

ALTERNATE EXAMPLE

Showing That a Function Is Not Differentiable

Show that $g(x) = 2|x+4|$ has no derivative at $x = -4$.

Solution

The function g is continuous for all real numbers and $g(-4) = 2|-4+4| = 0$. Using Form (1) of the definition of the derivative to find the two one-sided limits at -4,

$$\lim_{x \to -4^-} \frac{g(x)-g(-4)}{x-(-4)} = \lim_{x \to -4^-} \frac{2|x+4|-0}{x+4} = \lim_{x \to -4^-} \frac{2|x+4|}{x+4} = -2$$

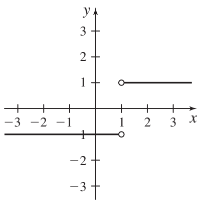
$$\lim_{x \to -4^+} \frac{g(x)-g(-4)}{x-(-4)} = \lim_{x \to -4^+} \frac{2|x+4|-0}{x+4} = \lim_{x \to -4^+} \frac{2|x+4|}{x+4} = 2$$

Since $\lim_{x \to -4^-} \frac{g(x)-g(-4)}{x-(-4)} = -2$, which is not equal to $\lim_{x \to -4^+} \frac{g(x)-g(-4)}{x-(-4)} = 2$, we conclude that the derivative does not exist at -4. The graph has a corner at the point $(-4, 0)$.

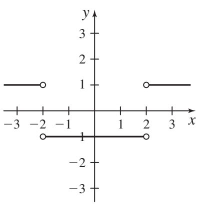
Teaching Tip

It can be a good idea to familiarize students with graphs in the form $\frac{|f(x)|}{f(x)}$. This graph consists of multiple horizontal line segments. Two examples are shown:

$$y = \frac{|x-1|}{x-1}$$



$$y = \frac{|x^2-4|}{x^2-4}$$



Notice that the function jumps at the zeros of $f(x)$. These functions can be useful when evaluating limits or asking questions about continuity.

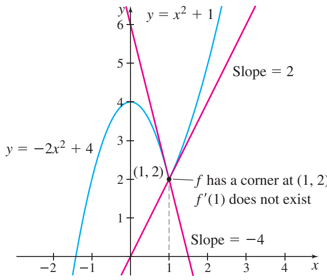


Figure 16 f has a corner at (1, 2).

If $x < 1$, then $f(x) = -2x^2 + 4$; if $x \ge 1$, then $f(x) = x^2 + 1$. So, it is necessary to find the one-sided limits at 1.

$$\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{(-2x^2+4)-2}{x-1} = \lim_{x \to 1^-} \frac{-2(x^2-1)}{x-1}$$
$$= -2 \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x-1} = -2 \lim_{x \to 1^-} (x+1) = -4$$
$$\lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^+} \frac{(x^2+1)-2}{x-1} = \lim_{x \to 1^+} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1^+} (x+1) = 2$$

Since the one-sided limits are not equal, $\lim_{x \to 1} \frac{f(x)-f(1)}{x-1}$ does not exist, and so $f'(1)$ does not exist. ■

Figure 16 illustrates the graph of the function f from Example 6. At 1, where the derivative does not exist, the graph of f has a corner. We usually say that the graph of f is not *smooth* at a corner.

NOW WORK Problem 39 and AP® Practice Problems 1, 5 and 9.

Example 7 illustrates the behavior of the graph of a function f when the derivative at a number c does not exist because $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is infinite.

CALC CLIP

EXAMPLE 7 Showing That a Function Is Not Differentiable

Show that $f(x) = (x-2)^{4/5}$ is not differentiable at 2.

Solution

The function f is continuous for all real numbers and $f(2) = (2-2)^{4/5} = 0$. Use Form (1) of the definition of the derivative to find the two one-sided limits at 2.

$$\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^-} \frac{(x-2)^{4/5}-0}{x-2} = \lim_{x \to 2^-} \frac{(x-2)^{4/5}}{x-2} = \lim_{x \to 2^-} \frac{1}{(x-2)^{1/5}} = -\infty$$
$$\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^+} \frac{(x-2)^{4/5}-0}{x-2} = \lim_{x \to 2^+} \frac{(x-2)^{4/5}}{x-2} = \lim_{x \to 2^+} \frac{1}{(x-2)^{1/5}} = \infty$$

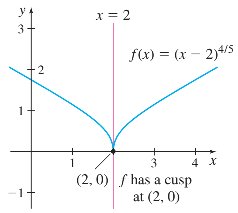


Figure 17 $f'(2)$ does not exist; the point (2, 0) is a cusp of the graph of f.

Since $\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = -\infty$ and $\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \infty$, we conclude that the function f is not differentiable at 2. The graph of f has a vertical tangent line at the point (2, 0), which is a cusp of the graph. See Figure 17. ■

NOW WORK Problem 35.

EXAMPLE 8 Obtaining Information about $y = f(x)$ from the Graph of Its Derivative Function

Suppose $y = f(x)$ is continuous for all real numbers. Figure 18 shows the graph of its derivative function f'.

(a) Does the graph of f have any horizontal tangent lines? If yes, explain why and identify where they occur.

(b) Does the graph of f have any vertical tangent lines? If yes, explain why, identify where they occur, and determine whether the point is a cusp of f.

(c) Does the graph of f have any corners? If yes, explain why and identify where they occur.

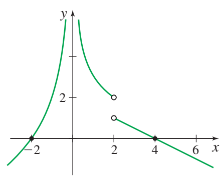


Figure 18 $y = f'(x)$

Solution

(a) Since the derivative f' equals the slope of a tangent line, horizontal tangent lines occur where the derivative equals 0. Since $f'(x) = 0$ for $x = -2$ and $x = 4$, the graph of f has two horizontal tangent lines, one at the point $(-2, f(-2))$ and the other at the point $(4, f(4))$.

(b) As x approaches 0, the derivative function f' approaches ∞ both for $x<0$ and for $x>0$. The graph of f has a vertical tangent line at $x=0$. The point $(0, f(0))$ is not a cusp because both limits equal ∞.

(c) The derivative is not defined at 2 but the one-sided derivatives have unequal finite limits as x approaches 2. So the graph of f has a corner at $(2, f(2))$. ■

NOW WORK Problem 45.

Differentiability and Continuity

NEED TO REVIEW? Continuity is discussed in Section 1.3, pp. 102–110.

In Chapter 1, we investigated the continuity of a function. Here we have been investigating the differentiability of a function. An important connection exists between continuity and differentiability.

THEOREM

If a function f is differentiable at a number c, then f is continuous at c.

Proof To show that f is continuous at c, we need to verify that $\lim_{x\to c} f(x)=f(c)$. We begin by observing that if $x\neq c$, then

$$f(x)-f(c)=\left[\frac{f(x)-f(c)}{x-c}\right](x-c)$$

We take the limit of both sides as $x\to c$, and use the fact that the limit of a product equals the product of the limits (we show later that each limit exists).

$$\lim_{x\to c}[f(x)-f(c)]=\lim_{x\to c}\left\{\left[\frac{f(x)-f(c)}{x-c}\right](x-c)\right\}=\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\left[\lim_{x\to c}(x-c)\right]$$

Since f is differentiable at c, we know that

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)$$

is a number. Also for any real number c, $\lim_{x\to c}(x-c)=0$. So

$$\lim_{x\to c}[f(x)-f(c)]=[f'(c)]\left[\lim_{x\to c}(x-c)\right]=f'(c)\cdot 0=0$$

That is, $\lim_{x\to c} f(x)=f(c)$, so f is continuous at c. ■

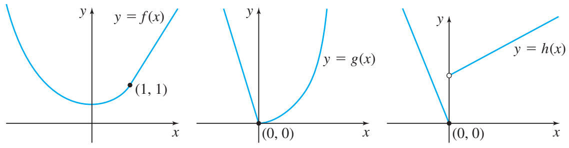
An equivalent statement of this theorem gives a condition under which a function has no derivative.

COROLLARY

If a function f is discontinuous at a number c, then f is not differentiable at c.

IN WORDS Differentiability implies continuity, but continuity does not imply differentiability.

Let's look at some of the possibilities. In Figure 19(a), the function f is continuous at the number 1 and has a derivative at 1. The function g, graphed in Figure 19(b), is continuous at the number 0, but it has no derivative at 0. So continuity at a number c provides no prediction about differentiability. On the other hand, the function h graphed in Figure 19(c) illustrates the corollary: If h is discontinuous at a number, it is not differentiable at that number.



(a) f is continuous at 1, and $f'(1)$ exists. (b) g is continuous at 0, but $g'(0)$ does not exist. (c) h is discontinuous at 0, so $h'(0)$ does not exist.

Figure 19

MATHEMATICAL PRACTICES TIP

Practice 3: Justification

You should build in activities where students must write justifications for their reasoning throughout the course. For example, ask students if a continuous function must be differentiable. Have students draw a function that is continuous but not differentiable. Have students write a justification to support their claim. (If students are having trouble coming up with an example on their own, draw the graph of $f(x)=|x|$ on the board and ask them to write a justification about why it is continuous but not differentiable.)

AP® EXAM TIP

Even though it is known that differentiability implies continuity, students must explicitly make this connection on free-response questions on the AP® Calculus Exam. If students are told that a function is differentiable and want to use a theorem that requires continuity, students must state that the function is continuous before using the theorem.

The corollary is useful if we are seeking the derivative of a function f that we suspect is discontinuous at a number c. If we can show that f is discontinuous at c, then the corollary affirms that the function f has no derivative at c. For example, since the floor function $f(x) = \lfloor x \rfloor$ is discontinuous at every integer c, it has no derivative at an integer.

EXAMPLE 9 Determining Whether a Function Is Differentiable at a Number

Determine whether the function

$$f(x) = \begin{cases} 2x+2 & \text{if } x<3 \\ 5 & \text{if } x=3 \\ x^2-1 & \text{if } x>3 \end{cases}$$

is differentiable at 3.

Solution

Since f is a piecewise-defined function, it may be discontinuous at 3 and therefore not differentiable at 3. So we begin by determining whether f is continuous at 3.

Since $f(3) = 5$, the function f is defined at 3. Use one-sided limits to check whether $\lim_{x \to 3} f(x)$ exists.

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x+2) = 8 \qquad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2-1) = 8$$

Since $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$, then $\lim_{x \to 3} f(x)$ exists. But $\lim_{x \to 3} f(x) = 8$ and $f(3) = 5$, so f is discontinuous at 3. From the corollary, since f is discontinuous at 3, the function f is not differentiable at 3. ■

Figure 20 shows the graph of f.

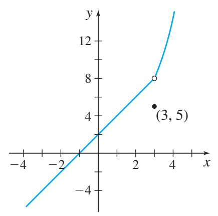


Figure 20 $f(x) = \begin{cases} 2x+2 & \text{if } x<3 \\ 5 & \text{if } x=3 \\ x^2-1 & \text{if } x>3 \end{cases}$

NOW WORK Problem 43.

In Example 9, the function f is discontinuous at 3, so by the corollary, the derivative of f at 3 does not exist. But when a function is continuous at a number c, then sometimes the derivative at c exists and other times the derivative at c does not exist.

EXAMPLE 10 Determining Whether a Function Is Differentiable at a Number

Determine whether each piecewise-defined function is differentiable at c. If the function has a derivative at c, find it.

(a) $f(x) = \begin{cases} x^3 & \text{if } x<0 \\ x^2 & \text{if } x \geq 0 \end{cases} \quad c = 0$ **(b)** $g(x) = \begin{cases} 1-2x & \text{if } x \leq 1 \\ x-2 & \text{if } x>1 \end{cases} \quad c = 1$

Solution

(a) See Figure 21. The function f is continuous at 0, which you should verify. To determine whether f has a derivative at 0, we examine the one-sided limits at 0 using Form (1).

For $x < 0$,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x^3 - 0}{x} = \lim_{x \to 0^-} x^2 = 0$$

For $x > 0$,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 - 0}{x} = \lim_{x \to 0^+} x = 0$$

Since both one-sided limits are equal, f is differentiable at 0, and $f'(0) = 0$.

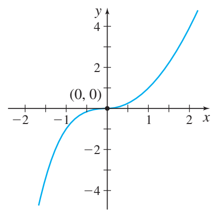


Figure 21 $f(x) = \begin{cases} x^3 & \text{if } x<0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

AP® CALC SKILL BUILDER FOR EXAMPLE 9

Determining If a Function Is Differentiable at a Number

If $f(x) = \begin{cases} \dfrac{x^2-64}{x-8} & \text{if } x \neq 8 \\ 8 & 8 \end{cases}$, which of the following statements about f are true?

I. $\lim_{x \to 8} f(x)$ exists.

II. f is continuous at $x = 8$.

III. f is differentiable at $x = 8$.

Solution

I. I is true because

$$\lim_{x \to 8} f(x) = \lim_{x \to 8} \frac{x^2-64}{x-8} = \lim_{x \to 8} \frac{(x-8)(x+8)}{x-8} = \lim_{x \to 8} x+8 = 16.$$

II. II is false because $\lim_{x^\circ\ 8} f(x) \neq f(8)$.

III. III is false because f is discontinuous at $x = 8$ and a discontinuous function is not differentiable.

AP® CALC SKILL BUILDER FOR EXAMPLE 9

Determining If a Function Is Differentiable or Continuous

Let f be a function such that $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = 4$. Which of the following statements must be true?

(a) f is continuous at $x = 2$.

(b) f is continuous at $x = 4$.

(c) f is differentiable at $x = 2$.

(d) f is differentiable at $x = 4$.

Solution

(a) True. The derivative of f at $x = 2$ is 4, so f must be continuous at $x = 2$. A function that is differentiable at a point is also continuous at that point.

(b) Not enough information. We do not know if f is continuous at $x = 4$.

(c) True. The limit statement tells us that the derivative of f at $x = 2$ is 4.

(d) Not enough information. We do not know if f is differentiable at $x = 4$.

ALTERNATE EXAMPLE

Determining Whether a Function Is Differentiable at a Number

Determine whether the piecewise-defined function has a derivative at c. If the function has a derivative at c, find it.

$$f(x) = \begin{cases} x^2 & \text{if } x<1 \\ 2x-1 & \text{if } x \geq 1 \end{cases} \quad c = 1$$

Solution

The function f is continuous at 1, which you should verify. To determine whether f has a derivative at 1, we examine the one-sided limits at 1 using Form (1).

For $x < 1$,

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{x^2-1}{x-1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1^-} x+1 = 2$$

For $x > 1$,

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{(2x-1)-1}{x-1} = \lim_{x \to 1^+} \frac{2x-2}{x-1} = \lim_{x^\circ\ 1^+} 2 = 2$$

Since both one-sided limits are equal, f has a derivative at 1 and $f'(1) = 2$.

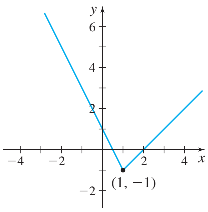


Figure 22 $g(x)=\begin{cases}1-2x & \text{if } x\le 1\\ x-2 & \text{if } x>1\end{cases}$

(b) See Figure 22. The function g is continuous at 1, which you should verify. To determine whether g is differentiable at 1, examine the one-sided limits at 1 using Form (1).

For $x<1$,

$$\lim_{x\to 1^-}\frac{g(x)-g(1)}{x-1}=\lim_{x\to 1^-}\frac{(1-2x)-(-1)}{x-1}=\lim_{x\to 1^-}\frac{2-2x}{x-1}$$
$$=\lim_{x\to 1^-}\frac{-2(x-1)}{x-1}=\lim_{x\to 1^-}(-2)=-2$$

For $x>1$,

$$\lim_{x\to 1^+}\frac{g(x)-g(1)}{x-1}=\lim_{x\to 1^+}\frac{(x-2)-(-1)}{x-1}=\lim_{x\to 1^+}\frac{x-1}{x-1}=1$$

The one-sided limits are not equal, so $\lim_{x\to 1}\frac{g(x)-g(1)}{x-1}$ does not exist. That is, g is not differentiable at 1. ■

Notice in Figure 21 the tangent lines to the graph of f turn smoothly around the origin. On the other hand, notice in Figure 22 the tangent lines to the graph of g change abruptly at the point $(1,-1)$, where the graph of g has a corner.

NOW WORK Problem 41 and AP® Practice Problems 3, 4, 6, and 7.

2.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The domain of a function f and the domain of its derivative function f' are always equal.
2. *True or False* If a function is continuous at a number c, then it is differentiable at c.
3. *Multiple Choice* If f is continuous at a number c and if $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ is infinite, then the graph of f has [**(a)** a horizontal **(b)** a vertical **(c)** no] tangent line at c.
4. The instruction, "Differentiate f," means to find the ____ of f.

Skill Building

In Problems 5–10, find the derivative of each function f at any real number c. Use Form (1) on page 171.

5. $f(x)=10$ **6.** $f(x)=-4$ **7.** $f(x)=2x+3$

8. $f(x)=3x-5$ **9.** $f(x)=2-x^2$ **10.** $f(x)=2x^2+4$

In Problems 11–16, differentiate each function f and determine the domain of f′. Use Form (2) on page 172.

11. $f(x)=5$ **12.** $f(x)=-2$

13. $f(x)=3x^2+x+5$ **14.** $f(x)=2x^2-x-7$

PAGE 173 **15.** $f(x)=5\sqrt{x-1}$ **16.** $f(x)=4\sqrt{x+3}$

In Problems 17–22, differentiate each function f. Graph $y=f(x)$ and $y=f'(x)$ on the same set of coordinate axes.

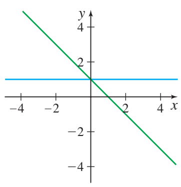
17. $f(x)=\frac{1}{3}x+1$ **18.** $f(x)=-4x-5$

PAGE 174 **19.** $f(x)=2x^2-5x$ **20.** $f(x)=-3x^2+2$

21. $f(x)=x^3-8x$ **22.** $f(x)=-x^3-8$

In Problems 23–26, for each figure determine if the graphs represent a function f and its derivative f′. If they do, indicate which is the graph of f and which is the graph of f′.

23.



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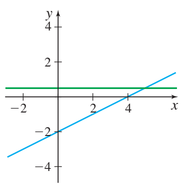

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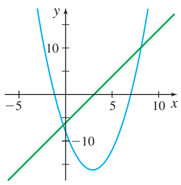

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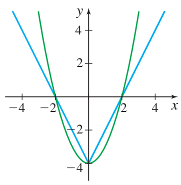


Answers to Section 2.2 Problems

1. False. **2.** False. **3.** **(b)** Vertical. **4.** Derivative. **5.** 0 **6.** 0 **7.** 2 **8.** 3 **9.** $-2c$ **10.** $4c$

11. $f'(x)=0$; all real numbers.

12. $f'(x)=0$; all real numbers.

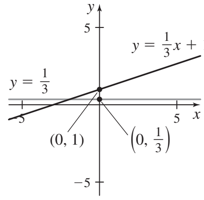
13. $f'(x)=6x+1$; all real numbers.

14. $f'(x)=4x-1$; all real numbers.

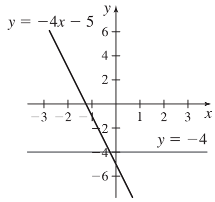
15. $f'(x)=\frac{5}{2\sqrt{x-1}}$; $x>1$

16. $f'(x)=\frac{2}{\sqrt{(x+3)}}$; $x>3$

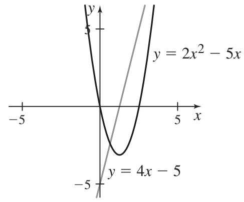
17. $f'(x)=\frac{1}{3}$



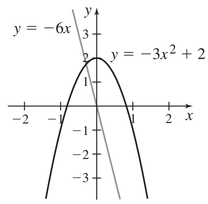
18. $f'(x)=-4$



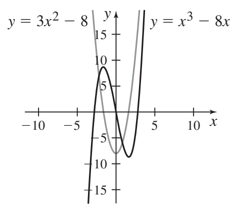
19. $f'(x)=4x-5$



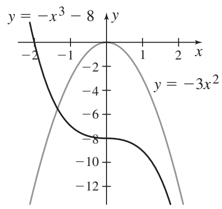
20. $f'(x)=-6x$



21. $f'(x)=3x^2-8$



22. $f'(x)=-3x^2$



Answers continue on p. 180

MUST-DO PROBLEMS FOR EXAM READINESS

AB: 13, 15, 19, 23, 25, 29, 31–34, 35, 39, 41, 43, 45, 47, 67, and all AP® Practice Problems

BC: 15, 19, 29–45 (odd), 49, 65, 67, 69, and all AP® Practice Problems

TRM Full Solutions to Section 2.2 Problems and AP® Practice Problems

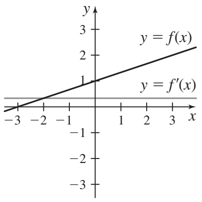
23. Not a graph of f and f'.

24. Graph of f (blue curve) and f' (green curve).

25. Graph of f (blue curve) and f' (green curve).

26. Not a graph of f and f'.

27.



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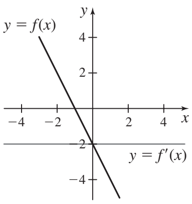

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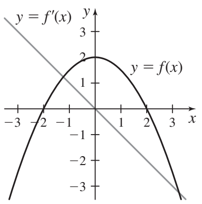

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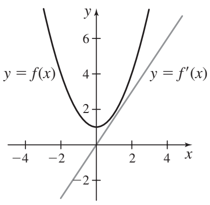


31. (B)

32. (D)

33. (A)

34. (C)

35. $f'(-8) = -\dfrac{1}{3}$

36. $f'(0)$ does not exist.

37. $f'(2)$ does not exist.

38. $f'(-2)$ does not exist.

39. $f'(1) = 2$

40. $f'(-1)$ does not exist.

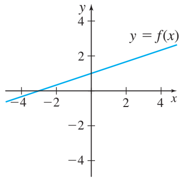
41. $f'\left(\dfrac{1}{2}\right)$ does not exist.

42. $f'(-1) = -4$

43. $f'(-1)$ does not exist.

In Problems 27–30, use the graph of f to obtain the graph of f′.

27.



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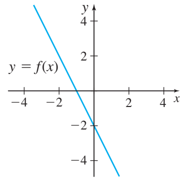
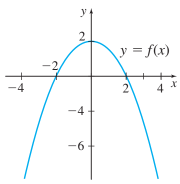


PAGE 175 **29.**



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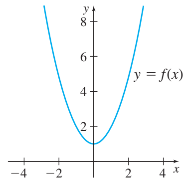
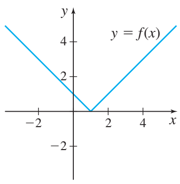


In Problems 31–34, the graph of a function f is given. Match each graph to the graph of its derivative f′ in A–D.

31.



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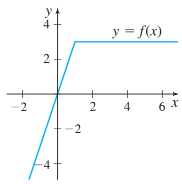

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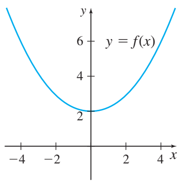

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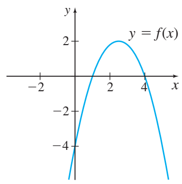
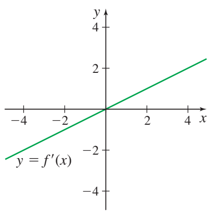
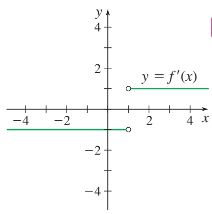
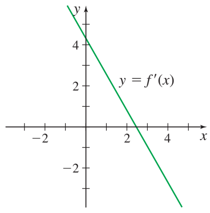




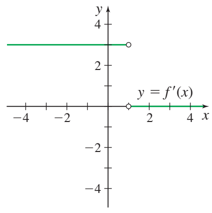
(A)



(B)



(C)



(D)

In Problems 35–44, determine whether each function f has a derivative at c. If it does, what is f′(c)? If it does not, give the reason why.

PAGE 176 **35.** $f(x) = x^{2/3}$ at $c = -8$

36. $f(x) = 2x^{1/3}$ at $c = 0$

37. $f(x) = |x^2 - 4|$ at $c = 2$

38. $f(x) = |x^2 - 4|$ at $c = -2$

PAGE 176 **39.** $f(x) = \begin{cases} 2x+3 & \text{if } x < 1 \\ x^2+4 & \text{if } x \geq 1 \end{cases}$ at $c = 1$

40. $f(x) = \begin{cases} 3-4x & \text{if } x < -1 \\ 2x+9 & \text{if } x \geq -1 \end{cases}$ at $c = -1$

PAGE 179 **41.** $f(x) = \begin{cases} -4+2x & \text{if } x \leq \frac{1}{2} \\ 4x^2-4 & \text{if } x > \frac{1}{2} \end{cases}$ at $c = \frac{1}{2}$

42. $f(x) = \begin{cases} 2x^2+1 & \text{if } x < -1 \\ -1-4x & \text{if } x \geq -1 \end{cases}$ at $c = -1$

PAGE 178 **43.** $f(x) = \begin{cases} 2x^2+1 & \text{if } x < -1 \\ 2+2x & \text{if } x \geq -1 \end{cases}$ at $c = -1$

44. $f(x) = \begin{cases} 5-2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ at $c = 2$

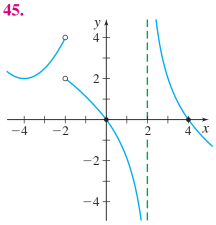
In Problems 45–48, each function f is continuous for all real numbers, and the graph of y = f′(x) is given.

(a) Does the graph of f have any horizontal tangent lines? If yes, explain why and identify where they occur.

(b) Does the graph of f have any vertical tangent lines? If yes, explain why, identify where they occur, and determine whether the point is a cusp of f.

(c) Does the graph of f have any corners? If yes, explain why and identify where they occur.

PAGE 177 **45.**



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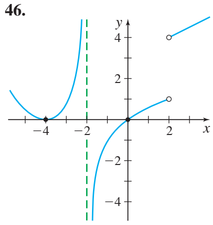
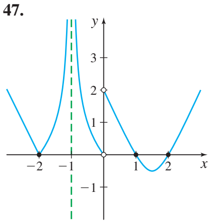


44. $f'(2)$ does not exist.

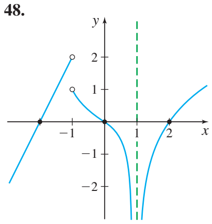
45. **(a)** Yes, at $x = 0$ and $x = 4$.
(b) Yes, a cusp at $x = 2$.
(c) Yes, at $x = -2$.

46. **(a)** Yes, at $x = -4$ and $x = 0$.
(b) Yes, a cusp at $x = -2$.
(c) Yes, at $x = 2$, a corner.

47.



48.

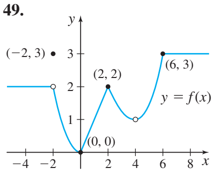


In Problems 49 and 50, use the given points $(c, f(c))$ on the graph of the function f.

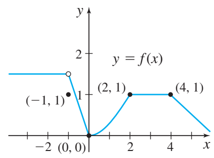
(a) For which numbers c does $\lim_{x\to c} f(x)$ exist but f is not continuous at c?

(b) For which numbers c is f continuous at c but not differentiable at c?

49.



50.



In Problems 51–54, find the derivative of each function.

51. $f(x) = mx + b$

52. $f(x) = ax^2 + bx + c$

53. $f(x) = \dfrac{1}{x^2}$

54. $f(x) = \dfrac{1}{\sqrt{x}}$

Applications and Extensions

In Problems 55–66, each limit represents the derivative of a function f at some number c. Determine f and c in each case.

55. $\lim_{h\to 0} \dfrac{(2+h)^2 - 4}{h}$

56. $\lim_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$

57. $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1}$

58. $\lim_{x\to 1} \dfrac{x^4 - 1}{x - 1}$

59. $\lim_{h\to 0} \dfrac{\sqrt{9+h} - 3}{h}$

60. $\lim_{h\to 0} \dfrac{(8+h)^{1/3} - 2}{h}$

61. $\lim_{x\to \pi/6} \dfrac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}}$

62. $\lim_{x\to \pi/4} \dfrac{\cos x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$

63. $\lim_{x\to 0} \dfrac{2(x+2)^2 - (x+2) - 6}{x}$

64. $\lim_{x\to 0} \dfrac{3x^3 - 2x}{x}$

65. $\lim_{h\to 0} \dfrac{(3+h)^2 + 2(3+h) - 15}{h}$

66. $\lim_{h\to 0} \dfrac{3(h-1)^2 + h - 3}{h}$

PAGE 173 **67. Units** The volume V (in cubic feet) of a balloon is expanding according to $V = V(t) = 4t$, where t is the time (in seconds). Find the rate of change of the volume of the balloon with respect to time. What are the units of $V'(t)$?

68. Units The area A (in square miles) of a circular patch of oil is expanding according to $A = A(t) = 2t$, where t is the time (in hours). At what rate is the area changing with respect to time? What are the units of $A'(t)$?

69. Units A manufacturer of precision digital switches has a daily cost C (in dollars) of $C(x) = 10{,}000 + 3x$, where x is the number of switches produced daily. What is the rate of change of cost with respect to x? What are the units of $C'(x)$?

70. Units A manufacturer of precision digital switches has daily revenue R (in dollars) of $R(x) = 5x - \dfrac{x^2}{2000}$, where x is the number of switches produced daily. What is the rate of change of revenue with respect to x? What are the units of $R'(x)$?

71. $f(x) = \begin{cases} x^3 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(a) Determine whether f is continuous at 0.
(b) Determine whether $f'(0)$ exists.
(c) Graph the function f and its derivative f'.

72. For the function $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(a) Determine whether f is continuous at 0.
(b) Determine whether $f'(0)$ exists.
(c) Graph the function f and its derivative f'.

73. Velocity The distance s (in feet) of an automobile from the origin at time t (in seconds) is given by the position function

$$s = s(t) = \begin{cases} t^3 & \text{if } 0 \le t < 5 \\ 125 & \text{if } t \ge 5 \end{cases}$$

(This could represent a crash test in which a vehicle is accelerated until it hits a brick wall at $t = 5$ s.)

(a) Find the velocity just before impact (at $t = 4.99$ s) and just after impact (at $t = 5.01$ s).
(b) Is the velocity function $v = s'(t)$ continuous at $t = 5$?
(c) How do you interpret the answer to (b)?

74. Population Growth A simple model for population growth states that the rate of change of population size P with respect to time t is proportional to the population size. Express this statement as an equation involving a derivative.

75. Atmospheric Pressure Atmospheric pressure p decreases as the distance x from the surface of Earth increases, and the rate of change of pressure with respect to altitude is proportional to the pressure. Express this law as an equation involving a derivative.

76. Electrical Current Under certain conditions, an electric current I will die out at a rate (with respect to time t) that is proportional to the current remaining. Express this law as an equation involving a derivative.

77. Tangent Line Let $f(x) = x^2 + 2$. Find all points on the graph of f for which the tangent line passes through the origin.

78. Tangent Line Let $f(x) = x^2 - 2x + 1$. Find all points on the graph of f for which the tangent line passes through the point $(1, -1)$.

47. (a) Yes, at $x = -2$, $x = 1$, $x = 2$.
(b) Yes, at $x = -1$, not a cusp.
(c) Yes, at $x = 0$.

48. (a) Yes, at $x = -2$, $x = 0$, and $x = 2$.
(b) Yes, at $x = 1$, not a cusp.
(c) Yes, at $x = -1$.

49. (a) -2 and 4
(b) 0, 2, 6

50. (a) -1
(b) 2, 4

51. $f'(x) = m$

52. $f'(x) = 2ax + b$

53. $f'(x) = -\dfrac{2}{x^3}$

54. $f'(x) = -\dfrac{1}{2x^{3/2}}$

55. $f(x) = x^2$, $c = 2$

56. $f(x) = x^3$, $c = 2$

57. $f(x) = x^2$, $c = 1$

58. $f(x) = x^4$, $c = 1$

59. $f(x) = \sqrt{x}$, $c = 9$

60. $f(x) = x^{\frac{1}{3}}$, $c = 8$

61. $f(x) = \sin x$, $c = \dfrac{\pi}{6}$

62. $f(x) = \cos x$, $c = \dfrac{\pi}{4}$

63. $f(x) = 2(x+2)^2 - (x+2)$, $c = 0$

64. $f(x) = 3x^3 - 2x$, $c = 0$

65. $f(x) = 3x^2 + 2x$, $c = 3$

66. $f(x) = 3x^2 + x$, $c = -1$

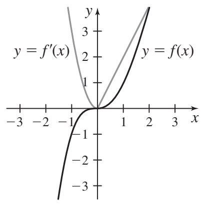
67. $V'(t) = 4$; ft^3/s

68. $A'(t) = 2$; mi^2/h

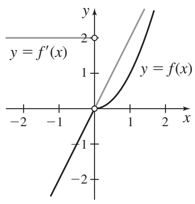
69. $C'(x) = 3$; dollars/switch

70. $R'(x) = 5 - \dfrac{x}{1000}$; dollars/switch

71. (a) Continuous at 0.
(b) $f'(0) = 0$
(c)



72. (a) Continuous at 0.
(b) $f'(0)$ does not exist.
(c)



73. (a) $s'(4.99) = 74.7003$ ft/s; $s'(5.01) = 0$ ft/s
(b) Not continuous.
(c) Answers will vary.

74. $P'(t) = kP$ for some k.

75. $p'(x) = -kp$, $k > 0$

76. $I'(t) = kI$, $k < 0$

77. $(\sqrt{2}, 4)$ and $(-\sqrt{2}, 4)$

78. (0, 1) and (2, 1)

79. (a) $2\pi r(° r) + \pi(° r)^2$
(b) $2\pi ° r$
(c) $2\pi r + \pi\,\Delta r$
(d) 2π
(e) 2π

80. (a) $4\pi r^2(° r) + 4\pi r(° r)^2 + \frac{4}{3}\pi(° r)^3$
(b) $4\pi r^2 + 4\pi r(\Delta r) + \frac{4}{3}\pi(\Delta r)^2$
(c) $4\pi r^2$

81. $\lim_{x\to 0^-} f'(x) = -1 \neq 1 = \lim_{x\to 0^+} f'(x)$

82. $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty$

83. See TSM for proof.

84. See TSM for proof.

85. (a) Parallel.
(b) Perpendicular.

86. (a) and (b) See TSM for proofs.

87. (a) $k = 4$
(b) $f'(3) = 12$
(c) $f'(x) = 4x$

88. See TSM for proof.

Answers to AP® Practice Problems

1. C
2. B
3. A
4. D
5. C
6. B
7. C
8. A
9. B
10. (a) $G'(5) = -30$
(b) At 5:00 a.m., oil is leaking out of the tank at the rate of 30 gal/h.
11. (a) -3
(b) 8 cm from the heated end, the temperature of the rod is changing at the rate of -3°C/cm.

79. Area and Circumference of a Circle A circle of radius r has area $A = \pi r^2$ and circumference $C = 2\pi r$. If the radius changes from r to $r + \Delta r$, find the:

(a) Change in area.
(b) Change in circumference.
(c) Average rate of change of area with respect to radius.
(d) Average rate of change of circumference with respect to radius.
(e) Rate of change of circumference with respect to radius.

80. Volume of a Sphere The volume V of a sphere of radius r is $V = \frac{4\pi r^3}{3}$. If the radius changes from r to $r + \Delta r$, find the:

(a) Change in volume.
(b) Average rate of change of volume with respect to radius.
(c) Rate of change of volume with respect to radius.

81. Use the definition of the derivative to show that $f(x) = |x|$ is not differentiable at 0.

82. Use the definition of the derivative to show that $f(x) = \sqrt[3]{x}$ is not differentiable at 0.

83. If f is an even function that is differentiable at c, show that its derivative function is odd. That is, show $f'(-c) = -f'(c)$.

84. If f is an odd function that is differentiable at c, show that its derivative function is even. That is, show $f'(-c) = f'(c)$.

85. Tangent Lines and Derivatives Let f and g be two functions, each with derivatives at c. State the relationship between their tangent lines at c if:

(a) $f'(c) = g'(c)$ (b) $f'(c) = -\frac{1}{g'(c)}$ $g'(c) \neq 0$

Challenge Problems

86. Let f be a function defined for all real numbers x. Suppose f has the following properties:

$$f(u+v) = f(u)f(v) \qquad f(0) = 1 \qquad f'(0) \text{ exists}$$

(a) Show that $f'(x)$ exists for all real numbers x.
(b) Show that $f'(x) = f'(0)f(x)$.

87. A function f is defined for all real numbers and has the following three properties:

$$f(1) = 5 \qquad f(3) = 21 \qquad f(a+b) - f(a) = kab + 2b^2$$

for all real numbers a and b where k is a fixed real number independent of a and b.

(a) Use $a = 1$ and $b = 2$ to find k.
(b) Find $f'(3)$.
(c) Find $f'(x)$ for all real x.

88. A function f is **periodic** if there is a positive number p so that $f(x+p) = f(x)$ for all x. Suppose f is differentiable. Show that if f is periodic with period p, then f' is also periodic with period p.

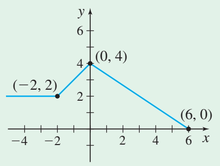
Preparing for the AP® Exam

AP® Practice Problems

PAGE 176 **1.** The function $f(x) = \begin{cases} x^2 - ax & \text{if } x \leq 1 \\ ax + b & \text{if } x > 1 \end{cases}$, where a and b are constants. If f is differentiable at $x = 1$, then $a + b =$

(A) -3 (B) -2 (C) 0 (D) 2

PAGE 172 **2.** The graph of the function f, given below, consists of three line segments. Find $\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$.



(A) -1 (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) does not exist

PAGE 179 **3.** If $f(x) = \begin{cases} \frac{x^2 - 25}{x - 5} & \text{if } x \neq 5 \\ 5 & \text{if } x = 5 \end{cases}$ which of the following statements about f are true?

I. $\lim_{x\to 5} f$ exists.
II. f is continuous at $x = 5$.
III. f is differentiable at $x = 5$.

(A) I only (B) I and II only
(C) I and III only (D) I, II, and III

PAGE 179 **4.** Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0) = 3$, $f(2) = -3$, and $f(7) = 3$, which of the following must be true?

I. f has at least 2 zeros.
II. f is continuous on the closed interval $[-1, 7]$.
III. For some c, $0 < c < 7$, $f(c) = -2$.

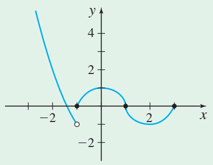
(A) I only (B) I and II only
(C) II and III only (D) I, II, and III

PAGE 176 **5.** If $f(x) = |x|$, which of the following statements about f are true?

I. f is continuous at 0.
II. f is differentiable at 0.
III. $f(0) = 0$.

(A) I only (B) III only
(C) I and III only (D) I, II, and III

PAGE 179 **6.** The graph of the function f shown in the figure has horizontal tangent lines at the points $(0, 1)$ and $(2, -1)$ and a vertical tangent line at the point $(1, 0)$. For what numbers x in the open interval $(-2, 3)$ is f not differentiable?



(A) -1 only (B) -1 and 1 only
(C) -1, 0, and 2 only (D) -1, 0, 1, and 2

PAGE 179 **7.** Let f be a function for which $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = -3$. Which of the following must be true?

I. f is continuous at 1.
II. f is differentiable at 1.
III. f' is continuous at 1.

(A) I only (B) II only
(C) I and II only (D) I, II, and III

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

(A) $(1, -3)$ (B) $(1, 2)$ (C) $(2, 0)$ (D) $(2, 4)$

PAGE 176 **9.** At $x = 2$, the function $f(x) = \begin{cases} 4x + 1 & \text{if } x \le 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.
(B) Continuous but not differentiable.
(C) Differentiable but not continuous.
(D) Neither continuous nor differentiable.

PAGE 173 **10.** Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.

(a) Find $G'(5)$ using the definition of the derivative.
(b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

PAGE 173 **11.** A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the table to approximate $T'(8)$.
(b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

OBJECTIVES *When you finish this section, you should be able to:*

1. **Differentiate a constant function (p. 184)**
2. **Differentiate a power function (p. 184)**
3. **Differentiate the sum and the difference of two functions (p. 186)**
4. **Differentiate the exponential function $y = e^x$ (p. 189)**

Finding the derivative of a function from the definition can become tedious, especially if the function f is complicated. Just as we did for limits, we derive some basic derivative formulas and some properties of derivatives that make finding a derivative simpler.

Before getting started, we introduce other notations commonly used for the derivative $f'(x)$ of a function $y = f(x)$. The most common ones are

$$y' \qquad \frac{dy}{dx} \qquad Df(x)$$

Leibniz notation $\frac{dy}{dx}$ may be written in several equivalent ways as

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$$

where $\frac{d}{dx}$ is an instruction to find the derivative (with respect to the independent variable x) of the function $y = f(x)$.

In **operator notation** $Df(x)$, D is said to *operate* on the function, and the result is the derivative of f. To emphasize that the operation is performed with respect to the independent variable x, it is sometimes written $Df(x) = D_x f(x)$.

We use prime notation or Leibniz notation, or sometimes a mixture of the two, depending on which is more convenient. We do not use the notation $Df(x)$ in this book.

Teaching Tip

Make connections between students' understanding of slope and the naming conventions for the derivative.

$$\frac{dy}{dx}$$

This ("derivative of y with respect to x") is equivalent to saying small change in y over small change in x, which is the slope.

$$\frac{d}{dx}y$$

This means the same thing as $\frac{dy}{dx}$, yet it can be read as an instruction. Take the derivative with respect to x of y.

$$\frac{d}{dx}f(x)$$

This also means the same thing as $\frac{dy}{dx}$, yet it specifically names the function that we wish to differentiate ("the derivative with respect to x of f(x)").

SUGGESTED SKILL 4.C

In an AP® Calculus course, students should always use correct notation. To help students learn how to correctly write mathematical statements, ensure that you are using proper notation whenever you model work for students. It can also be helpful to regularly use different notations (such as y' or $\frac{dy}{dx}$) so that students feel comfortable seeing and using both.

TRM Alternate Examples Section 2.3

You can find the Alternate Examples for this section in PDF format in the Teacher's Resource Materials.

TRM AP® Calc Skill Builders Section 2.3

You can find the AP® Calc Skill Builders for this section in PDF format in the Teacher's Resource Materials.

COMMON ERRORS & MISCONCEPTIONS

Once students learn the derivative rules, they sometimes forget that π and e are constants. For example, $\frac{d}{dx}(\pi^2) = 0$, not 2π.

Teaching Tip

When you are teaching students derivative rules in Chapters 2 and 3, work through the proof of each rule on the board with students when it is introduced. This will help students get practice with the limit definition of a derivative and help them to see how the rules they are learning are connected to the definition of a derivative they learned in Section 2.1.

Teaching Tip

If you took an abbreviated approach with Sections 2.1 and 2.2 and focused only on using the definition of a derivative to find the derivative of a function, remember to introduce the terminology "instantaneous rate of change" as the students learn the derivative rules.

AP® EXAM TIP

Derivative questions often appear in the multiple-choice section of the AP® Exam. Approximately three or four of the multiple-choice questions tend to come directly from this chapter. Students will be prepared for all multiple-choice derivative questions upon the completion of Chapter 3.

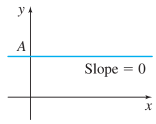


Figure 23 $f(x) = A$

1 Differentiate a Constant Function

See Figure 23. Since the graph of a constant function $f(x) = A$ is a horizontal line, the tangent line to f at any point is also a horizontal line, whose slope is 0. Since the derivative is the slope of the tangent line, the derivative of f is 0.

THEOREM Derivative of a Constant Function

If f is the constant function $f(x) = A$, then

$$\boxed{f'(x) = 0}$$

That is, if A is a constant, then

$$\boxed{\frac{d}{dx}A = 0}$$

IN WORDS The derivative of a constant is 0.

Proof If $f(x) = A$, then its derivative function is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{A - A}{h} = 0$$

↑ The definition of a derivative, Form (2)   ↑ $f(x) = A$, $f(x+h) = A$ ■

EXAMPLE 1 Differentiating a Constant Function

(a) If $f(x) = \sqrt{3}$, then $f'(x) = 0$ **(b)** If $f(x) = -\frac{1}{2}$, then $f'(x) = 0$

(c) If $f(x) = \pi$, then $\frac{d}{dx}\pi = 0$ **(d)** If $f(x) = 0$, then $\frac{d}{dx}0 = 0$ ■

2 Differentiate a Power Function

Next we analyze the derivative of a power function $f(x) = x^n$, where $n \geq 1$ is an integer.

When $n = 1$, then $f(x) = x$ is the identity function and its graph is the line $y = x$, as shown in Figure 24.

The slope of the line $y = x$ is 1, so we would expect $f'(x) = 1$.

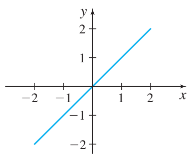


Figure 24 $f(x) = x$

Proof $$f'(x) = \frac{d}{dx}x = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h) - x}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

↑ $f(x) = x$, $f(x+h) = x + h$ ■

THEOREM Derivative of $f(x) = x$

If $f(x) = x$, then

$$\boxed{f'(x) = \frac{d}{dx}x = 1}$$

When $n = 2$, then $f(x) = x^2$ is the square function. The derivative of f is

$$f'(x) = \frac{d}{dx}x^2 = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x + h) = 2x$$

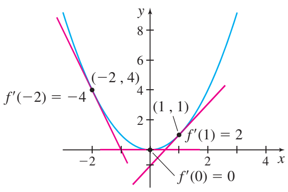


Figure 25 $f(x) = x^2$

The slope of the tangent line to the graph of $f(x) = x^2$ is different for every number x. Figure 25 shows the graph of f and several of its tangent lines. Notice that the slope of each tangent line drawn is twice the value of x.

COMMON ERRORS & MISCONCEPTIONS

Students sometimes confuse functions of the form $y = x^n$ with $y = a^x$. For example, $y = x^4$ can be differentiated using the Simple Power Rule, whereas $y = 4^x$ is exponential, so the Simple Power Rule does not apply. The Simple Power Rule only applies to functions of the form $y = x^n$.

When $n = 3$, then $f(x) = x^3$ is the cube function. The derivative of f is

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$$

Notice that the derivative of each of these power functions is another power function, whose degree is 1 less than the degree of the original function and whose coefficient is the degree of the original function. This rule holds for all power functions as the following theorem, called the *Simple Power Rule*, indicates.

IN WORDS The derivative of x raised to an integer power $n \geq 1$ is n times x raised to the power $n - 1$.

THEOREM Simple Power Rule

The derivative of the power function $y = x^n$, where $n \geq 1$ is an integer, is

$$y' = \frac{d}{dx}x^n = nx^{n-1}$$

NEED TO REVIEW? The Binomial Theorem is discussed in Section P.8, pp. 72–73.

Proof If $f(x) = x^n$ and n is a positive integer, then $f(x+h) = (x+h)^n$. We use the Binomial Theorem to expand $(x+h)^n$. Then

$$f(x+h) = (x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$
$$= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n}{h} \quad \text{Simplify.}$$
$$= \lim_{h\to 0}\frac{h\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}\right]}{h} \quad \text{Factor } h \text{ in the numerator.}$$
$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}\right] \quad \text{Divide out the common } h.$$
$$= nx^{n-1} \quad \text{Take the limit. Only the first term remains.}$$

■

NOTE $\frac{d}{dx}x^n = nx^{n-1}$ is true not only for positive integers n but also for any real number n. But the proof requires future results. As these are developed, we will expand the Power Rule to include an ever-widening set of numbers until we arrive at the fact it is true when n is a real number.

EXAMPLE 2 Differentiating a Power Function

(a) $\frac{d}{dx}x^5 = 5x^4$ **(b)** If $g(x) = x^{10}$, then $g'(x) = 10x^9$. ■

NOW WORK Problem 1 and AP® Practice Problem 1.

But what if we want to find the derivative of the function $f(x) = ax^n$ when $a \neq 1$? The next theorem, called the *Constant Multiple Rule*, provides a way.

AP® CALC SKILL BUILDER FOR EXAMPLE 2

Differentiating a Power Function

Find each derivative.

(a) $\frac{d}{dx}x^3$

(b) $\frac{d}{dx}x^4$

(c) $\frac{d}{dx}x^7$

(d) $\frac{d}{dx}x^{18}$

Solution

(a) $\frac{d}{dx}x^3 = 3x^2$

(b) $\frac{d}{dx}x^4 = 4x^3$

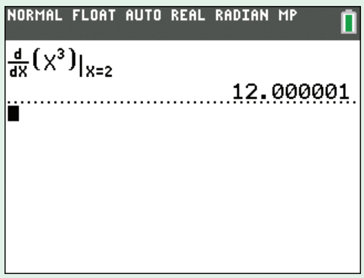
(c) $\frac{d}{dx}x^7 = 7x^6$

(d) $\frac{d}{dx}x^{18} = 18x^{17}$

GRAPHING CALCULATOR PRACTICE

Use your calculator to find the derivative of $y = x^3$ at $x = 2$.

Solution



According to the calculator, the derivative is 12.000001, but the actual answer is 12. (You can have your students verify this with the Power Rule.) If you round the calculator answer to the third digit after the decimal place, then you get the correct answer of 12.

Teaching Tip

Students will learn many versions of the Power Rule throughout the course that extend the rule to include all real values of n. Even though these have not been introduced yet, it can be helpful to extend the rule for students now, so that they can practice taking the derivative of functions like $y = \frac{1}{x} = x^{-1}$ (Power Rule for any integer n in Section 2.4) and $y = \sqrt{x} = x^{1/2}$ (Power Rule for rational exponents in Section 3.2).

COMMON ERRORS & MISCONCEPTIONS

Students often trust answers that their calculators give them more than their own work. This can lead to difficulties because the calculator usually relies on an algorithm to approximate a solution rather than find the correct answer. The Graphing Calculator Practice on this page illustrates how the numeric derivative feature on the calculator can sometimes give an answer that is not perfect.

COMMON ERRORS & MISCONCEPTIONS

Students may not recognize that $f(x)=\frac{x}{2}$ is the same as $f(x)=\frac{1}{2}x$. These expressions are equivalent. The derivative is $f'(x)=\frac{1}{2}$.

ALTERNATE EXAMPLE

Differentiating a Constant Times a Power Function

Find the derivative of each function.

(a) $f(x)=4x^6$ **(b)** $g(x)=\frac{x^5}{10}$

(c) $v(t)=2et^2$

Solution

(a) $f(x)=4^{\circ}x^6$, so

$$f'(x)=4\left[\frac{d}{dx}x^6\right]=4\cdot 6x^5=24x^5$$

(b) $g(x)=\frac{1}{10}{}^{\circ}x^5$, so

$$g'(x)=\frac{1}{10}\cdot\left[\frac{d}{dx}x^5\right]=\frac{1}{10}\cdot 5x^4=\frac{x^4}{2}$$

(c) $v(t)=2e^{\circ}t^2$, so

$$v'(t)=2e\cdot\left[\frac{d}{dt}t^2\right]=2e\cdot 2t=4et$$

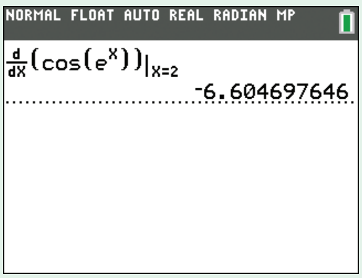
Building Calculator Skills

Consider having students use the numerical derivative feature on a calculator to practice this skill for the AP® Exam. Students can use a calculator to verify their answers or you can ask students to use their calculators to differentiate functions they do not have a rule for yet.

GRAPHING CALCULATOR PRACTICE

Use your calculator to find the derivative of $y=\cos(e^x)$ at $x=2$.

Solution



At $x=2$, the derivative is –6.605.

IN WORDS The derivative of a constant times a differentiable function f equals the constant times the derivative of f.

THEOREM Constant Multiple Rule

If a function f is differentiable and k is a constant, then $F(x)=kf(x)$ is a function that is differentiable and

$$F'(x)=kf'(x)$$

Proof Use the definition of a derivative, Form (2).

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$$

$$=\lim_{h\to 0}\frac{k[f(x+h)-f(x)]}{h}=k\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=k\cdot f'(x)$$ ■

Using Leibniz notation, the Constant Multiple Rule takes the form

$$\frac{d}{dx}[kf(x)]=k\left[\frac{d}{dx}f(x)\right]$$

A change in the symbol used for the independent variable does not affect the derivative formula. For example, $\frac{d}{dt}t^2=2t$ and $\frac{d}{du}u^5=5u^4$.

EXAMPLE 3 Differentiating a Constant Times a Power Function

Find the derivative of each function:

(a) $f(x)=5x^3$ **(b)** $g(u)=-\frac{1}{2}u^2$ **(c)** $u(x)=\pi^4x^3$

Solution

Notice that each of these functions involves the product of a constant and a power function. So, we use the Constant Multiple Rule followed by the Simple Power Rule.

(a) $f(x)=5\cdot x^3$, so $f'(x)=5\left[\frac{d}{dx}x^3\right]=5\cdot 3x^2=15x^2$

(b) $g(u)=-\frac{1}{2}\cdot u^2$, so $g'(u)=-\frac{1}{2}\cdot\frac{d}{du}u^2=-\frac{1}{2}\cdot 2u^1=-u$

(c) $u(x)=\pi^4x^3$, so $u'(x)=\pi^4\cdot\frac{d}{dx}x^3=\pi^4\cdot 3x^2=3\pi^4x^2$ ■
(π is a constant)

NOW WORK Problem 31.

3 Differentiate the Sum and the Difference of Two Functions

We can find the derivative of a function that is the sum of two functions whose derivatives are known by adding the derivatives of each function.

IN WORDS The derivative of the sum of two differentiable functions equals the sum of their derivatives. That is, $(f+g)'=f'+g'$.

THEOREM Sum Rule

If two functions f and g are differentiable and if $F(x)=f(x)+g(x)$, then F is differentiable and

$$F'(x)=f'(x)+g'(x)$$

AP® EXAM TIP

On free-response questions, all nonexact solutions must be reported to three decimal places. The answer provided may be rounded to three decimal places or truncated after three decimal places.

Teaching Tip

As the students are learning the derivative rules, take advantage of every opportunity to strengthen their algebra skills. Notice the AP® Calc Skill Builder directly above. We could have stopped with $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}+2$. When the students learn about the First Derivative Test, the Second Derivative Test, and optimization, they will have to use the derivative in rational form: $f'(x)=\frac{1+4\sqrt{x}}{2\sqrt{x}}$. Each of the forms presented helps in different situations. Consider requiring the students to simplify at this point in the year.

Proof If $F(x) = f(x) + g(x)$, then

$$F(x+h) - F(x) = [f(x+h) + g(x+h)] - [f(x) + g(x)]$$
$$= [f(x+h) - f(x)] + [g(x+h) - g(x)]$$

So, the derivative of F is

$$F'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

The limit of a sum is the sum of the limits. ■

In Leibniz notation, the Sum Rule takes the form

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

EXAMPLE 4 Differentiating the Sum of Two Functions

Find the derivative of $f(x) = 3x^2 + 8$.

Solution

Here f is the sum of $3x^2$ and 8. So, we begin by using the Sum Rule.

$$f'(x) = \frac{d}{dx}(3x^2 + 8) = \frac{d}{dx}(3x^2) + \frac{d}{dx}8 = 3\frac{d}{dx}x^2 + 0 = 3 \cdot 2x = 6x$$ ■

(Sum Rule; Constant Multiple Rule; Simple Power Rule)

NOW WORK Problem 7 and AP® Practice Problem 6.

THEOREM Difference Rule

If the functions f and g are differentiable and if $F(x) = f(x) - g(x)$, then F is differentiable, and $F'(x) = f'(x) - g'(x)$. That is,

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

IN WORDS The derivative of the difference of two differentiable functions is the difference of their derivatives. That is, $(f - g)' = f' - g'$.

The proof of the Difference Rule is left as an exercise. (See Problem 78.)

The Sum and Difference Rules extend to sums (or differences) of more than two functions. That is, if the functions $f_1, f_2, \ldots, f_n$ are all differentiable, and $a_1, a_2, \ldots, a_n$ are constants, then

$$\frac{d}{dx}[a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)] = a_1 \frac{d}{dx} f_1(x) + a_2 \frac{d}{dx} f_2(x) + \cdots + a_n \frac{d}{dx} f_n(x)$$

Combining the rules for finding the derivative of a constant, a power function, and a sum or difference allows us to differentiate any polynomial function.

EXAMPLE 5 Differentiating a Polynomial Function

(a) Find the derivative of $f(x) = 2x^4 - 6x^2 + 2x - 3$.
(b) What is $f'(2)$?
(c) Find the slope of the tangent line to the graph of f at the point $(1, -5)$.
(d) Find an equation of the tangent line to the graph of f at the point $(1, -5)$.
(e) Find an equation of the normal line to the graph of f at the point $(1, -5)$.
(f) Use technology to graph f, the tangent line, and the normal line to the graph of f at the point $(1, -5)$ on the same screen.

AP® CALC SKILL BUILDER FOR EXAMPLE 5

Differentiating a Polynomial Function

Given the function $f(x) = 2x^5 - 3x^2$, find the following:

(a) $f'(x)$
(b) $f'(-2)$
(c) What is the slope of the tangent line to the graph of f at the point $(1, -1)$?
(d) Find an equation of the tangent line to the graph of f at the point $(1, -1)$.
(e) Find an equation of the normal line to the graph of f at the point $(1, -1)$.
(f) Use technology to graph f, the tangent line, and the normal line to the graph of f at the point $(1, -1)$ on the same screen.

Solution

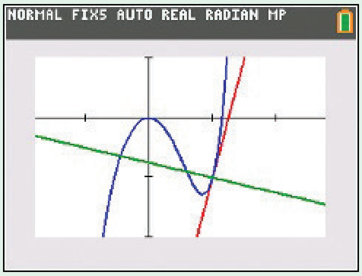
(a) $f'(x) = 10x^4 - 6x$
(b) $f'(-2) = 10(-2)^4 - 6(-2) = 172$
(c) The slope of the tangent line at the point $(1, -1)$ is $f'(1) = 10(1)^4 - 6(1) = 4$.
(d) Use the point-slope form of an equation of a line to find an equation of the tangent line at $(1, -1)$.

$$y - (-1) = 4(x - 1)$$
$$y = 4x - 5$$

(e) Since the normal line and the tangent line at the point $(1, -1)$ on the graph of f are perpendicular and the slope of the tangent line is 4, the slope of the normal line is $-\frac{1}{4}$.

$$y - (-1) = -\frac{1}{4}(x - 1)$$
$$y = -\frac{1}{4}x - \frac{3}{4}$$

(f) The graphs of f, the tangent line, and the normal line to f at $(1, -1)$ are shown.



AP® CALC SKILL BUILDER FOR EXAMPLE 4

Differentiating the Sum of Functions

Find the derivative of $f(x) = \sqrt{x} + 2x + 3$.

Solution

Rewrite the function:

$$f(x) = x^{\frac{1}{2}} + 2x + 3$$

Differentiate:

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 2$$

which can be re-written as:

$$f'(x) = \frac{1}{2\sqrt{x}} + 2$$

or as:

$$f'(x) = \frac{1 + 4\sqrt{x}}{2\sqrt{x}}$$

TRM Section 2.3: Worksheet 1

This worksheet contains 3 polynomial functions and their graphs. The students are asked to find the derivative of each function, to find the points where the graph of each function has a horizontal tangent line, and to examine the graph to determine if their answers are reasonable.

Teaching Tip

Example 6 is an excellent precursor for many topics coming up in Chapter 4, such as finding the maximum and minimum of a function, local extrema, concavity, using calculus to graph functions, and optimization.

ALTERNATE EXAMPLE

Solving Equations and Inequalities Involving Derivatives

Find *x*-coordinates of the points on the graph of $f(x)=\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{3}{2}x^2$ where f has a horizontal tangent line.

Solution

We want to find the numbers x for which $f'(x)=0$.

$$f'(x)=x^3+2x^2-3x$$
$$f'(x)=x(x^2+2x-3)$$
$$f'(x)=x(x+3)(x-1)$$

$f'(x)=0$ when $x=0$, $x=-3$, and $x=1$. The graph of f has a horizontal tangent line at $x=0$, $x=-3$, and $x=1$.

Solution

(a) $f'(x)=\frac{d}{dx}(2x^4-6x^2+2x-3)=\frac{d}{dx}(2x^4)-\frac{d}{dx}(6x^2)+\frac{d}{dx}(2x)-\frac{d}{dx}3$ ← Sum/Difference Rules

$=2\cdot\frac{d}{dx}x^4-6\cdot\frac{d}{dx}x^2+2\cdot\frac{d}{dx}x-0$ ← Constant Multiple Rule

$=2\cdot 4x^3-6\cdot 2x+2\cdot 1=8x^3-12x+2$ ← Simple Power Rule; Simplify

(b) $f'(2)=8\cdot 2^3-12\cdot 2+2=64-24+2=42$.

(c) The slope of the tangent line at the point $(1,\,-5)$ equals $f'(1)$.

$$f'(1)=8\cdot 1^3-12\cdot 1+2=8-12+2=-2$$

(d) Use the point-slope form of an equation of a line to find an equation of the tangent line at $(1,\,-5)$.

$$y-(-5)=-2(x-1)$$
$$y=-2(x-1)-5=-2x+2-5=-2x-3$$

The line $y=-2x-3$ is tangent to the graph of $f(x)=2x^4-6x^2+2x-3$ at the point $(1,\,-5)$.

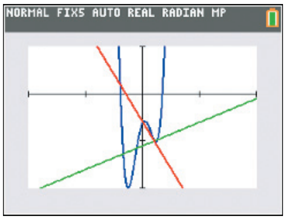
(e) Since the normal line and the tangent line at the point $(1,\,-5)$ on the graph of f are perpendicular and the slope of the tangent line is -2, the slope of the normal line is $\frac{1}{2}$.

Use the point-slope form of an equation of a line to find an equation of the normal line.

$$y-(-5)=\frac{1}{2}(x-1)$$
$$y=\frac{1}{2}(x-1)-5=\frac{1}{2}x-\frac{1}{2}-5=\frac{1}{2}x-\frac{11}{2}$$

The line $y=\frac{1}{2}x-\frac{11}{2}$ is normal to the graph of f at the point $(1,\,-5)$.

(f) The graphs of f, the tangent line, and the normal line to f at $(1,\,-5)$ are shown in Figure 26. ■



$[-10, 10]\times[-10, 5]$

Figure 26 $f(x)=2x^4-6x^2+2x-3$

NOW WORK Problem 33 and AP® Practice Problems 2, 5, 10, and 11.

In some applications, we need to solve equations or inequalities involving the derivative of a function.

EXAMPLE 6 Solving Equations and Inequalities Involving Derivatives

(a) Find the points on the graph of $f(x)=4x^3-12x^2+2$, where f has a horizontal tangent line.

(b) Where is $f'(x)>0$? Where is $f'(x)<0$?

Solution

(a) The slope of a horizontal tangent line is 0. Since the derivative of f equals the slope of the tangent line, we need to find the numbers x for which $f'(x)=0$.

$$f'(x)=12x^2-24x=12x(x-2)$$

$12x(x-2)=0$ $f'(x)=0$

$x=0$ or $x=2$ Solve.

At the points $(0,\,f(0))=(0,\,2)$ and $(2,\,f(2))=(2,\,-14)$, the graph of the function $f(x)=4x^3-12x^2+2$ has horizontal tangent lines.

BIG IDEAS TIP

Big Idea 3: Analysis of Functions

Using a function's derivative to determine its behavior is one of the major consequences of calculus. We can use this information to identify special points on a function, such as maximums and minimums. In Chapter 5, students will learn a variety of skills that will allow them to graph most functions without the aid of a calculator.

TRM Section 2.3: Worksheet 2

This worksheet is great for building up skill at taking the AP® Exam. The students are given information in a table, and then later in a graph, and are asked several derivative-based questions.

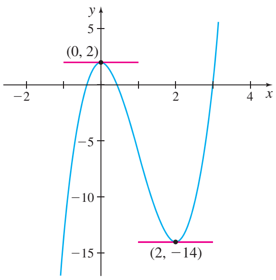


Figure 27 $f(x) = 4x^3 - 12x^2 + 2$

(b) Since $f'(x) = 12x(x-2)$ and we want to solve the inequalities $f'(x) > 0$ and $f'(x) < 0$, we use the zeros of f', 0 and 2, and form a table using the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$. ■

TABLE 2

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sign of $f'(x) = 12x(x-2)$	Positive	Negative	Positive

We conclude $f'(x) > 0$ on $(-\infty, 0) \cup (2, \infty)$ and $f'(x) < 0$ on $(0, 2)$.

Figure 27 shows the graph of f and the two horizontal tangent lines.

NOW WORK Problem 37 and AP® Practice Problem 3.

4 Differentiate the Exponential Function $y = e^x$

None of the differentiation rules developed so far allow us to find the derivative of an exponential function. To differentiate $f(x) = a^x$, we need to return to the definition of a derivative.

NEED TO REVIEW? Exponential functions are discussed in Section P.5, pp. 41–44.

We begin by making some general observations about the derivative of $f(x) = a^x$, $a > 0$ and $a \neq 1$. We then use these observations to find the derivative of the exponential function $y = e^x$.

Suppose $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The derivative of f is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \underset{a^{x+h}=a^x\cdot a^h}{=} \lim_{h\to 0}\frac{a^x\cdot a^h - a^x}{h}$$

$$\underset{\text{Factor out } a^x.}{=} \lim_{h\to 0}\left[a^x\cdot\frac{a^h-1}{h}\right] = a^x\cdot\lim_{h\to 0}\frac{a^h-1}{h}$$

provided $\lim_{h\to 0}\frac{a^h-1}{h}$ exists.

Three observations about the derivative of $f(x) = a^x$ are significant:

- $f'(0) = a^0 \lim_{h\to 0}\frac{a^h-1}{h} = \lim_{h\to 0}\frac{a^h-1}{h}$.
- $f'(x)$ is a multiple of a^x. In fact, $\frac{d}{dx}a^x = f'(0)\cdot a^x$.
- If $f'(0)$ exists, then $f'(x)$ exists, and the domain of f' is the same as that of $f(x) = a^x$, all real numbers.

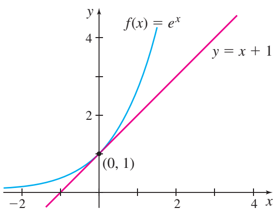


Figure 28

NEED TO REVIEW? The number e is discussed in Section P.5, pp. 44–45.

The slope of the tangent line to the graph of $f(x) = a^x$ at the point $(0, 1)$ is $f'(0) = \lim_{h\to 0}\frac{a^h-1}{h}$, and the value of this limit depends on the base a. In Section P.5, the number e was defined as that number for which the slope of the tangent line to the graph of $y = a^x$ at the point $(0, 1)$ equals 1. That is, if $f(x) = e^x$, then $f'(0) = 1$ so that

$$\boxed{\lim_{h\to 0}\frac{e^h-1}{h} = 1}$$

Figure 28 shows $f(x) = e^x$ and the tangent line $y = x + 1$ with slope 1 at the point $(0, 1)$.

Building Calculator Skills

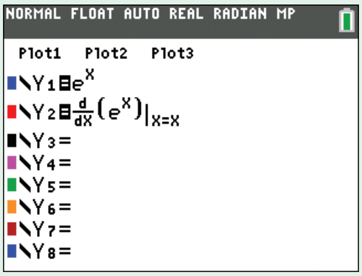
When introducing the derivative of a new function, consider having students use their calculators first to graph the function itself and then see if they can make a conjecture about what the derivative of the new function might be, before they also graph the derivative on the same set of axes.

GRAPHING CALCULATOR PRACTICE

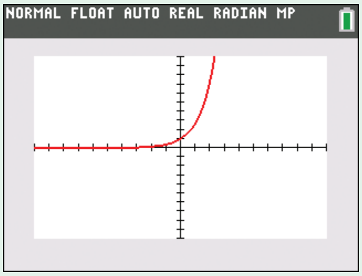
On your calculator, graph both the function $y = e^x$ and its derivative. Use the graphs to make a conjecture about a derivative rule for $y = e^x$.

Solution

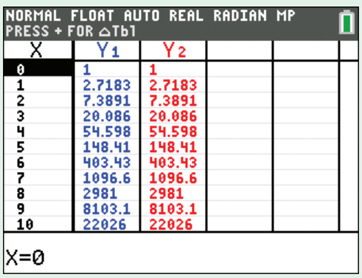
Graph both the function $y = e^x$ and its derivative.



As students watch both functions graph, they should notice that the derivative graph perfectly covers up the function.



Students can use a table to verify that both the function and derivative have the same numerical value for any given value of x.


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl

X	Y1	Y2
0	1	1
1	2.7183	2.7183
2	7.3891	7.3891
3	20.086	20.086
4	54.598	54.598
5	148.41	148.41
6	403.43	403.43
7	1096.6	1096.6
8	2981	2981
9	8103.1	8103.1
10	22026	22026

X=0

Thus, the derivative of $y = e^x$ is $y' = e^x$.

Since $\frac{d}{dx}a^x = f'(0)\cdot a^x$, if $f(x)=e^x$, then $\frac{d}{dx}e^x = f'(0)\cdot e^x = 1\cdot e^x = e^x$.

THEOREM Derivative of the Exponential Function $y = e^x$

The derivative of the exponential function $y = e^x$ is

$$y' = \frac{d}{dx}e^x = e^x \qquad (1)$$

CALC CLIP

EXAMPLE 7 Differentiating an Expression Involving $y = e^x$

Find the derivative of $f(x) = 4e^x + x^3$.

Solution

The function f is the sum of $4e^x$ and x^3. Then

$$f'(x) = \frac{d}{dx}(4e^x + x^3) = \frac{d}{dx}(4e^x) + \frac{d}{dx}x^3 = 4\frac{d}{dx}e^x + 3x^2 = 4e^x + 3x^2$$

↑ Sum Rule; ↑ Constant Multiple Rule; Simple Power Rule; ↑ Use (1). ■

NOTE We have not forgotten $y = \ln x$. Here is its derivative:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Use this result for now. We do not have the necessary mathematics to prove it until Chapter 3.

NOW WORK Problem 25 and AP® Practice Problems 4 and 9.

Now we know $\frac{d}{dx}e^x = e^x$. To find the derivative of $f(x) = a^x$, $a > 0$ and $a \neq 1$, we need more information. See Chapter 3.

2.3 Assess Your Understanding

Concepts and Vocabulary

PAGE 185 **1.** $\frac{d}{dx}\pi^2 =$ ____; $\frac{d}{dx}x^3 =$ ____.

2. When n is a positive integer, the Simple Power Rule states that $\frac{d}{dx}x^n =$ ____.

3. *True or False* The derivative of a power function of degree greater than 1 is also a power function.

4. If k is a constant and f is a differentiable function, then $\frac{d}{dx}[kf(x)] =$ ____.

5. The derivative of $f(x) = e^x$ is ____.

6. *True or False* The derivative of an exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is always a constant multiple of a^x.

Skill Building

In Problems 7–26, find the derivative of each function using the formulas of this section. (a, b, c, and d, when they appear, are constants.)

PAGE 187 **7.** $f(x) = 3x + \sqrt{2}$ **8.** $f(x) = 5x - \pi$

9. $f(x) = x^2 + 3x + 4$ **10.** $f(x) = 4x^4 + 2x^2 - 2$

11. $f(u) = 8u^5 - 5u + 1$ **12.** $f(u) = 9u^3 - 2u^2 + 4u + 4$

13. $f(s) = as^3 + \frac{3}{2}s^2$ **14.** $f(s) = 4 - \pi s^2$

15. $f(t) = \frac{1}{3}(t^5 - 8)$ **16.** $f(x) = \frac{1}{5}(x^7 - 3x^2 + 2)$

17. $f(t) = \frac{t^3+2}{5}$ **18.** $f(x) = \frac{x^7 - 5x}{9}$

19. $f(x) = \frac{x^3+2x+1}{7}$ **20.** $f(x) = \frac{1}{a}(ax^2+bx+c),\ a \neq 0$

21. $f(x) = ax^2 + bx + c$ **22.** $f(x) = ax^3 + bx^2 + cx + d$

23. $f(x) = 4e^x$ **24.** $f(x) = -\frac{1}{2}e^x$

PAGE 190 **25.** $f(u) = 5u^2 - 2e^u$ **26.** $f(u) = 3e^u + 10$

In Problems 27–32, find each derivative.

27. $\frac{d}{dt}\left(\sqrt{3}\,t + \frac{1}{2}\right)$ **28.** $\frac{d}{dt}\left(\frac{2t^4 - 5}{8}\right)$

29. $\frac{dA}{dR}$ if $A(R) = \pi R^2$ **30.** $\frac{dC}{dR}$ if $C = 2\pi R$

PAGE 186 **31.** $\frac{dV}{dr}$ if $V = \frac{4}{3}\pi r^3$ **32.** $\frac{dP}{dT}$ if $P = 0.2T$

In Problems 33–36:

(a) Find the slope of the tangent line to the graph of each function f at the indicated point.

(b) Find an equation of the tangent line at the point.

(c) Find an equation of the normal line at the point.

(d) Graph f and the tangent line and normal line found in (b) and (c) on the same set of axes.

PAGE 188 **33.** $f(x) = x^3 + 3x - 1$ at $(0, -1)$

34. $f(x) = x^4 + 2x - 1$ at $(1, 2)$

35. $f(x) = e^x + 5x$ at $(0, 1)$

36. $f(x) = 4 - e^x$ at $(0, 3)$

AP® CALC SKILL BUILDER FOR EXAMPLE 7

Differentiating an Expression Involving $y = e^x$

Find the rate of change of the function $f(x) = -2e^x + \frac{1}{4}x^2$ at $x = 6$.

Solution

The rate of change of f at $x = 6$ is $f'(6)$:

$$f'(x) = -2e^x + \frac{1}{2}x$$

$$f'(6) = -2e^6 + 3$$

TRM Section 2.3: Worksheet 3

This worksheet contains 10 problems in which the student is asked to find the derivative of a function.

MUST-DO PROBLEMS FOR EXAM READINESS

AB: 7, 11, 15, 19, 25, 29, 31, 35, 37, 43, 47, and all AP® Practice Problems

BC: 1, 7, 19, 25, 31, 33, 37–47 (odd), 53, 63, and all AP® Practice Problems

TRM Full Solutions to Section 2.3 Problems and AP® Practice Problems

Answers to Section 2.3 Problems

1. 0; $3x^2$ **2.** nx^{n-1}

3. True. **4.** $k\left[\frac{d}{dx}f(x)\right]$

5. e^x **6.** True.

7. $f'(x) = 3$ **8.** $f'(x) = 5$

9. $f'(x) = 2x + 3$ **10.** $f'(x) = 16x^3 + 4x$

11. $f'(u) = 40u^4 - 5$

12. $f'(u) = 27u^2 - 4u + 4$

13. $f'(s) = 3as^2 + 3s$

14. $f'(s) = -2\pi s$ **15.** $f'(t) = \frac{5}{3}t^4$

16. $f'(x) = \frac{7}{5}x^6 - \frac{6}{5}x$ **17.** $f'(t) = \frac{3}{5}t^2$

18. $f'(x) = \frac{7}{9}x^6 - \frac{5}{9}$

19. $f'(x) = \frac{3x^2+2}{7}$

20. $f'(x) = \frac{1}{a}(2ax + b)$

21. $f'(x) = 2ax + b$

22. $f'(x) = 3ax^2 + 2bx + c$

23. $f'(x) = 4e^x$

24. $f'(x) = -\frac{1}{2}e^x$

25. $f'(u) = 10u - 2e^u$

26. $f'(u) = 3e^u$

27. $\sqrt{3}$ **28.** t^3

29. $2\pi R$ **30.** 2π

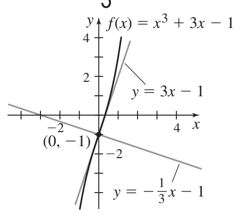
31. $4\pi r^2$ **32.** 0.2

33. (a) 3

(b) $y = 3x - 1$

(c) $y = -\frac{1}{3}x - 1$

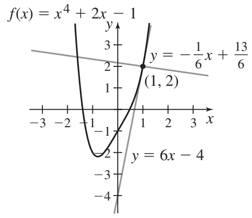
(d)



34. (a) 6 **(b)** $y = 6x - 4$

(c) $y = -\frac{1}{6}x + \frac{13}{6}$

(d)



35. (a) 6

(b) $y = 6x + 1$

(c) $y = -\frac{1}{6}x + 1$

Answers continue on p. 191

In Problems 37–42:

(a) *Find the points, if any, at which the graph of each function f has a horizontal tangent line.*

(b) *Find an equation for each horizontal tangent line.*

(c) *Solve the inequality* $f'(x) > 0$.

(d) *Solve the inequality* $f'(x) < 0$.

(e) *Graph f and any horizontal lines found in (b) on the same set of axes.*

(f) *Describe the graph of f for the results obtained in parts (c) and (d).*

37. $f(x) = 3x^2 - 12x + 4$ **38.** $f(x) = x^2 + 4x - 3$

39. $f(x) = x + e^x$ **40.** $f(x) = 2e^x - 1$

41. $f(x) = x^3 - 3x + 2$ **42.** $f(x) = x^4 - 4x^3$

43. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^3 - t + 1$. Find the velocity of the object at $t = 0$ and at $t = 5$.

44. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^4 - t^3 + 1$. Find the velocity of the object at $t = 0$ and at $t = 1$.

Rectilinear Motion *In Problems 45 and 46, each position function gives the signed distance s from the origin at time t of an object in rectilinear motion:*

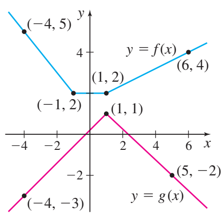
(a) *Find the velocity v of the object at any time t.*

(b) *When is the velocity of the object 0?*

45. $s(t) = 2 - 5t + t^2$ **46.** $s(t) = t^3 - \frac{9}{2}t^2 + 6t + 4$

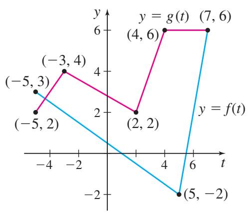
In Problems 47 and 48, use the graphs to find each derivative.

47. Let $u(x) = f(x) + g(x)$ and $v(x) = f(x) - g(x)$.



(a) $u'(0)$ **(b)** $u'(4)$
(c) $v'(-2)$ **(d)** $v'(6)$
(e) $3u'(5)$ **(f)** $-2v'(3)$

48. Let $F(t) = f(t) + g(t)$ and $G(t) = g(t) - f(t)$.



(a) $F'(0)$ **(b)** $F'(3)$
(c) $F'(-4)$ **(d)** $G'(-2)$
(e) $G'(-1)$ **(f)** $G'(6)$

In Problems 49 and 50, for each function f:

(a) *Find* $f'(x)$ *by expanding* $f(x)$ *and differentiating the polynomial.*

(b) *Find* $f'(x)$ *using a CAS.*

(c) *Show that the results found in parts (a) and (b) are equivalent.*

49. $f(x) = (2x - 1)^3$ **50.** $f(x) = (x^2 + x)^4$

Applications and Extensions

In Problems 51–56, find each limit.

51. $\lim\limits_{h\to 0}\dfrac{5\left(\frac{1}{2}+h\right)^8 - 5\left(\frac{1}{2}\right)^8}{h}$ **52.** $\lim\limits_{h\to 0}\dfrac{6(2+h)^5 - 6\cdot 2^5}{h}$

53. $\lim\limits_{h\to 0}\dfrac{\sqrt{3}(8+h)^5 - \sqrt{3}\cdot 8^5}{h}$ **54.** $\lim\limits_{h\to 0}\dfrac{\pi(1+h)^{10} - \pi}{h}$

55. $\lim\limits_{h\to 0}\dfrac{a(x+h)^3 - ax^3}{h}$ **56.** $\lim\limits_{h\to 0}\dfrac{b(x+h)^n - bx^n}{h}$

In Problems 57–62, find an equation of the tangent line(s) to the graph of the function f that is (are) parallel to the line L.

57. $f(x) = 3x^2 - x$; $L: y = 5x$

58. $f(x) = 2x^3 + 1$; $L: y = 6x - 1$

59. $f(x) = e^x$; $L: y - x - 5 = 0$

60. $f(x) = -2e^x$; $L: y + 2x - 8 = 0$

61. $f(x) = \frac{1}{3}x^3 - x^2$; $L: y = 3x - 2$

62. $f(x) = x^3 - x$; $L: x + y = 0$

63. Tangent Lines Let $f(x) = 4x^3 - 3x - 1$.

(a) Find an equation of the tangent line to the graph of f at $x = 2$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = x + 12$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = x + 12$, and any tangent lines found in (c) on the same screen.

64. Tangent Lines Let $f(x) = x^3 + 2x^2 + x - 1$.

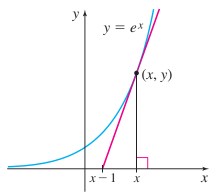
(a) Find an equation of the tangent line to the graph of f at $x = 0$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = 3x - 2$.

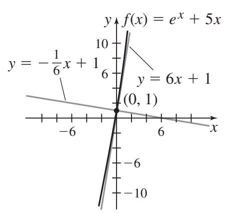
(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = 3x - 2$, and any tangent lines found in (c) on the same screen.

65. Tangent Line Show that the line perpendicular to the x-axis and containing the point (x, y) on the graph of $y = e^x$ and the tangent line to the graph of $y = e^x$ at the point (x, y) intersect the x-axis 1 unit apart. See the figure.



(d)

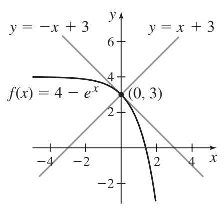


36. **(a)** -1

(b) $y = -x + 3$

(c) $y = -x + 3$

(d)



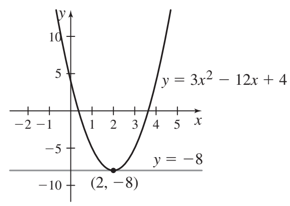
37. **(a)** $(2, -8)$

(b) $y = -8$

(c) $x > 2$

(d) $x < 2$

(e)

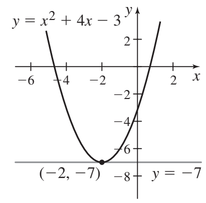


(f) Increasing when $x > 2$, decreasing when $x < 2$.

38. **(a)** $(-2, -7)$ **(b)** $y = -7$

(c) $x > -2$ **(d)** $x < -2$

(e)



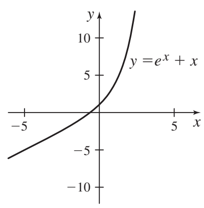
(f) Increasing when $x > -2$, decreasing when $x < -2$.

39. **(a)** None. **(b)** None.

(c) All real numbers.

(d) No solutions.

(e)



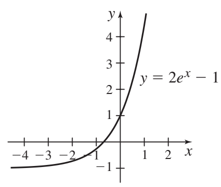
(f) Increasing for all x.

40. **(a)** None. **(b)** None.

(c) All real numbers.

(d) No solutions.

(e)

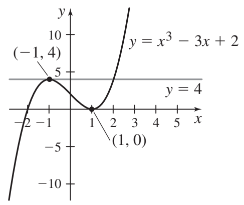


(f) Increasing for all x.

41. **(a)** $(1, 0)$, $(-1, 4)$ **(b)** $y = 0$, $y = 4$

(c) $x < -1$ or $x > 1$ **(d)** $-1 < x < 1$

(e)



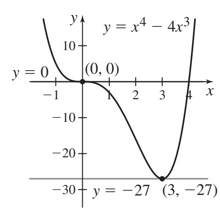
(f) Increasing when $x < -1$ or $x > 1$, decreasing when $-1 < x < 1$.

42. **(a)** $(0, 0)$ and $(3, -27)$

(b) $y = 0$, $y = -27$

(c) $x > 3$ **(d)** $x < 3$

(e)



Answers continue on p. 192

66. Tangent Line Show that the tangent line to the graph of $y = x^n$, $n \geq 2$ an integer, at $(1, 1)$ has y-intercept $1 - n$.

67. Tangent Lines If n is an odd positive integer, show that the tangent lines to the graph of $y = x^n$ at $(1, 1)$ and at $(-1, -1)$ are parallel.

68. Tangent Line If the line $3x - 4y = 0$ is tangent to the graph of $y = x^3 + k$ in the first quadrant, find k.

69. Tangent Line Find the constants a, b, and c so that the graph of $y = ax^2 + bx + c$ contains the point $(-1, 1)$ and is tangent to the line $y = 2x$ at $(0, 0)$.

70. Tangent Line Let T be the tangent line to the graph of $y = x^3$ at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$. At what other point Q on the graph of $y = x^3$ does the line T intersect the graph? What is the slope of the tangent line at Q?

71. Military Tactics A dive bomber is flying from right to left along the graph of $y = x^2$. When a rocket bomb is released, it follows a path that is approximately along the tangent line. Where should the pilot release the bomb if the target is at $(1, 0)$?

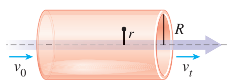
72. Military Tactics Answer the question in Problem 71 if the plane is flying from right to left along the graph of $y = x^3$.

73. Fluid Dynamics The velocity v of a liquid flowing through a cylindrical tube is given by the **Hagen–Poiseuille equation** $v = k(R^2 - r^2)$, where R is the radius of the tube, k is a constant that depends on the length of the tube and the velocity of the liquid at its ends, and r is the variable distance of the liquid from the center of the tube. See the figure below.

(a) Find the rate of change of v with respect to r at the center of the tube.

(b) What is the rate of change halfway from the center to the wall of the tube?

(c) What is the rate of change at the wall of the tube?



74. Rate of Change Water is leaking out of a swimming pool that measures 20 ft by 40 ft by 6 ft. The amount of water in the pool at a time t is $W(t) = 35{,}000 - 20t^2$ gallons, where t equals the number of hours since the pool was last filled. At what rate is the water leaking when $t = 2$ h?

75. Luminosity of the Sun The luminosity L of a star is the rate at which it radiates energy. This rate depends on the temperature T and surface area A of the star's photosphere (the gaseous surface that emits the light). Luminosity is modeled by the equation $L = \sigma AT^4$, where σ is a constant known as the **Stefan–Boltzmann constant**, and T is expressed in the absolute (Kelvin) scale for which 0 K is absolute zero. As with most stars, the Sun's temperature has gradually increased over the 6 billion years of its existence, causing its luminosity to slowly increase.

(a) Find the rate at which the Sun's luminosity changes with respect to the temperature of its photosphere. Assume that the surface area A remains constant.

(b) Find the rate of change at the present time. The temperature of the photosphere is currently 5800 K (10,000 °F), the radius of the photosphere is $r = 6.96 \times 10^8$ m, and $\sigma = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\ \text{K}^4}$.

(c) Assuming that the rate found in (b) remains constant, how much would the luminosity change if its photosphere temperature increased by 1 K (1 °C or 1.8 °F)? Compare this change to the present luminosity of the Sun.

76. Medicine: Poiseuille's Equation The French physician Poiseuille discovered that the volume V of blood (in cubic centimeters per unit time) flowing through an artery with inner radius R (in centimeters) can be modeled by

$$V(R) = kR^4$$

where $k = \frac{\pi}{8\nu l}$ is constant (here ν represents the viscosity of blood and l is the length of the artery).

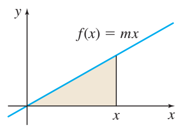
(a) Find the rate of change of the volume V of blood flowing through the artery with respect to the radius R.

(b) Find the rate of change when $R = 0.03$ and when $R = 0.04$.

(c) If the radius of a partially clogged artery is increased from 0.03 to 0.04 cm, estimate the effect on the rate of change of the volume V with respect to R of the blood flowing through the enlarged artery.

(d) How do you interpret the results found in (b) and (c)?

77. Derivative of an Area Let $f(x) = mx$, $m > 0$. Let $F(x)$, $x > 0$, be defined as the area of the shaded region in the figure. Find $F'(x)$.



78. The Difference Rule Prove that if f and g are differentiable functions and if $F(x) = f(x) - g(x)$, then

$$F'(x) = f'(x) - g'(x)$$

79. Simple Power Rule Let $f(x) = x^n$, where n is a positive integer. Use a factoring principle to show that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = nc^{n-1}$$

80. Normal Lines For what nonnegative number b is the line given by $y = -\frac{1}{3}x + b$ normal to the graph of $y = x^3$?

81. Normal Lines Let N be the normal line to the graph of $y = x^2$ at the point $(-2, 4)$. At what other point Q does N meet the graph?

43. $v(0) = -1$ m/s and $v(5) = 74$ m/s

44. $v(0) = 0$ m/s and $v(1) = 1$ m/s

45. (a) $v(t) = 2t - 5$ **(b)** $t = \frac{5}{2}$

46. (a) $v(t) = 3t^2 - 9t + 6$ **(b)** $t = 1$ or $t = 2$

47. (a) $\frac{4}{5}$ **(b)** $-\frac{7}{20}$ **(c)** $-\frac{9}{5}$ **(d)** $\frac{23}{20}$ **(e)** $-\frac{21}{20}$ **(f)** $-\frac{23}{10}$

48. (a) $-\frac{9}{10}$ **(b)** $\frac{3}{2}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{10}$ **(e)** $\frac{1}{10}$ **(f)** -4

49. (a) $f'(x) = 24x^2 - 24x + 6$ **(b)** $f'(x) = 6(2x - 1)^2$ **(c)** The expression in (b) expands to the expression in (a).

50. (a) $f'(x) = 8x^7 + 28x^6 + 36x^5 + 20x^4 + 4x^3$ **(b)** $f'(x) = 4x^3(2x + 1)(x + 1)^3$ **(c)** The expression in (b) expands to the expression in (a).

51. $\frac{5}{16}$ **52.** 480 **53.** $20{,}480\sqrt{3}$

54. 10π **55.** $3ax^2$ **56.** bnx^{n-1}

57. $y = 5x - 3$ **58.** $y = 6x - 3$, $y = 6x + 5$

59. $y = x + 1$ **60.** $y = -2x - 2$

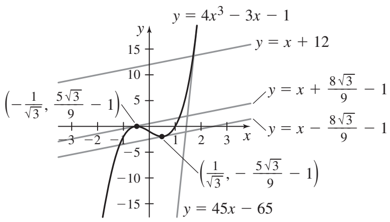
61. $y = 3x + \frac{5}{3}$, $y = 3x - 9$ **62.** $y = -x$

63. (a) $y = 45x - 65$

(b) $\left(\frac{\sqrt{3}}{3}, -\frac{5\sqrt{3}}{9} - 1\right)$, $\left(-\frac{\sqrt{3}}{3}, \frac{5\sqrt{3}}{9} - 1\right)$

(c) At $x = \frac{\sqrt{3}}{3}$, $y = x - \frac{8\sqrt{3}}{9} - 1$, and at $x = -\frac{\sqrt{3}}{3}$, $y = x + \frac{8\sqrt{3}}{9} - 1$

(d)

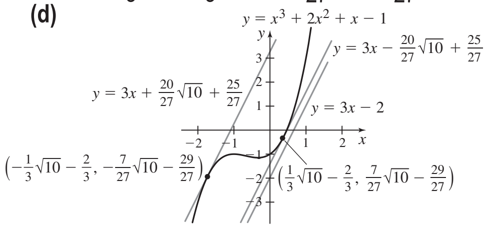


64. (a) $y = x - 1$

(b) $\left(\frac{1}{3}\sqrt{10} - \frac{2}{3}, \frac{7}{27}\sqrt{10} - \frac{29}{27}\right)$ and $\left(-\frac{1}{3}\sqrt{10} - \frac{2}{3}, -\frac{7}{27}\sqrt{10} - \frac{29}{27}\right)$

(c) At $x = \frac{1}{3}\sqrt{10} - \frac{2}{3}$, $y = 3x - \frac{20}{17}\sqrt{10} + \frac{25}{27}$, and at $x = -\frac{1}{3}\sqrt{10} - \frac{2}{3}$, $y = 3x + \frac{20}{27}\sqrt{10} + \frac{25}{27}$

(d)



65. See TSM for proof.

66. See TSM for proof.

67. See TSM for proof. **68.** $k = \frac{1}{4}$

69. $a = 3, b = 2, c = 0$ **70.** $(-1, -1)$, $\frac{3}{4}$

71. $(2, 4)$ **72.** $\left(\frac{3}{2}, \frac{27}{8}\right)$

73. (a) 0 **(b)** $-kR$ **(c)** $-2kR$

74. −80 gal/h

75. (a) $\frac{dL}{dt} = 4\sigma AT^3$ **(b)** $2.694 \times 10^{23}\ \frac{\text{W}}{\text{K}}$

Answers continue on p. 193

Challenge Problems

82. Tangent Line Find a, b, c, d so that the tangent line to the graph of the cubic $y = ax^3 + bx^2 + cx + d$ at the point $(1, 0)$ is $y = 3x - 3$ and at the point $(2, 9)$ is $y = 18x - 27$.

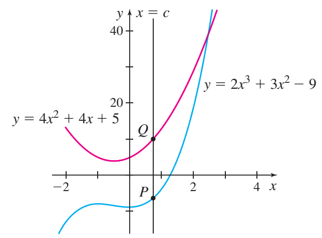
83. Tangent Line Find the fourth degree polynomial that contains the origin and to which the line $x + 2y = 14$ is tangent at both $x = 4$ and $x = -2$.

84. Tangent Lines Find equations for all the lines containing the point $(1, 4)$ that are tangent to the graph of $y = x^3 - 10x^2 + 6x - 2$. At what points do each of the tangent lines touch the graph?

85. The line $x = c$, where $c > 0$, intersects the cubic $y = 2x^3 + 3x^2 - 9$ at the point P and intersects the parabola $y = 4x^2 + 4x + 5$ at the point Q, as shown in the figure on the right.

(a) If the line tangent to the cubic at the point P is parallel to the line tangent to the parabola at the point Q, find the number c.

(b) Write an equation for each of the two tangent lines described in (a).



86. $f(x) = Ax^2 + B$, $A > 0$.

(a) Find c, $c > 0$, in terms of A so that the tangent lines to the graph of f at $(c, f(c))$ and $(-c, f(-c))$ are perpendicular.

(b) Find the slopes of the tangent lines in (a).

(c) Find the coordinates, in terms of A and B, of the point of intersection of the tangent lines in (a).

85. (a) $c = 1$

(b) $y = 12x - 16$ and $y = 12x + 1$

86. (a) $c = \frac{1}{2A}$

(b) 1 and –1

(c) $\left(0, -\frac{1}{4A} + B\right)$

Preparing for the AP® Exam

AP® Practice Problems

PAGE 185 **1.** If $g(x) = x$, then $g'(7) =$

(A) 0 (B) 1 (C) 7 (D) $\frac{49}{2}$

PAGE 188 **2.** The line $x + y = k$, where k is a constant, is a tangent line to the graph of the function $f(x) = x^2 - 5x + 2$. What is the value of k?

(A) -1 (B) 2 (C) -2 (D) -4

PAGE 189 **3.** An object moves along the x-axis so that its position at time t is $x(t) = 3t^2 - 9t + 7$. For what time t is the velocity of the object zero?

(A) -3 (B) 3 (C) $\frac{3}{2}$ (D) 7

PAGE 190 **4.** If $f(x) = e^x$, then $\ln(f'(3)) =$

(A) 3 (B) 0 (C) e^3 (D) $\ln 3$

PAGE 188 **5.** An equation of the normal line to the graph of $g(x) = x^3 + 2x^2 - 2x + 1$ at the point where $x = -2$ is

(A) $x + 2y = 12$ (B) $x - 2y = 8$
(C) $2x + y = -9$ (D) $x + 2y = 8$

PAGE 187 **6.** The line $9x - 16y = 0$ is tangent to the graph of $f(x) = 3x^3 + k$, where k is a constant, at a point in the first quadrant. Find k.

(A) $\frac{3}{32}$ (B) $\frac{3}{16}$ (C) $\frac{3}{64}$ (D) $\frac{9}{64}$

7. If $f(x) = 1 + |x - 4|$, find $f'(4)$.

(A) -1 (B) 0 (C) 1 (D) $f'(4)$ does not exist.

PAGE 187 **8.** The cost C (in dollars) of manufacturing x units of a product is $C(x) = 0.3x^2 + 4.02x + 3500$. What is the rate of change of C when $x = 1000$ units?

(A) 307.52 (B) 0.60402 (C) 604.02 (D) 1020

PAGE 190 **9.** $\frac{d}{dx}(5 \ln x) =$

(A) $\frac{1}{5x}$ (B) $5e^x$ (C) $-\frac{5}{\ln x}$ (D) $\frac{5}{x}$

PAGE 188 **10.** For the function $f(x) = x^2 + 4$

(a) Find $f'(1)$.

(b) Find an equation of the tangent line to the graph of f at $x = 1$.

(c) Find $f'(-4)$.

(d) Find an equation of the tangent line to the graph of f at $x = -4$.

(e) Find the point of intersection of the two tangent lines found in (b) and (d).

PAGE 188 **11.** Which is an equation of the tangent line to the graph of $f(x) = x^4 + 3x^2 + 2$ at the point where $f'(x) = 2$?

(A) $y = 2x + 2$ (B) $y = 2x + 2.929$
(C) $y = 2x + 1.678$ (D) $y = 2x - 2.929$

Answers to AP® Practice Problems

1. B
2. C
3. C
4. A
5. D
6. A
7. D
8. C
9. D
10. (a) 2; **(b)** $y = 2x + 3$; **(c)** –8; **(d)** $y = -8x - 12$; **(e)** $(-3/2, 0)$
11. C

(c) 2.694×10^{23} W

76. (a) $\frac{dV}{dt} = 4kR^3 \text{ cm}^3/\text{cm}$

(b) $V'(0.03) = 1.08k \times 10^{-4} \text{ cm}^3/\text{cm}$; $V'(0.04) = 256k \times 10^{-4} \text{ cm}^3/\text{cm}$

(c) 137% increase.

(d) Answers will vary. Sample answer: A $33\frac{1}{3}$% increase in the radius of the artery, from 0.03 to 0.04 cm, produces a 137% increase in the rate of change of volume flow.

77. $F'(x) = mx$

78. See TSM for proof.

79. See TSM for proof.

80. $b = \frac{4}{3}$ **81.** $Q = \left(\frac{9}{4}, \frac{81}{16}\right)$

82. $a = 3, b = -6, c = 6, d = -3$

83. $y = -\frac{7}{64}x^4 + \frac{7}{16}x^3 + \frac{21}{16}x^2 - 4x$

84. $y = 6x - 2$ at $(0, -2)$, $y = -\frac{101}{4}x + \frac{117}{4}$ at $\left(\frac{5}{2}, -\frac{271}{8}\right)$, and $y = -26x + 30$ at $(4, -74)$

COMMON ERRORS & MISCONCEPTIONS

Students often mistakenly believe that the derivative of a product is the product of the derivatives $\left(\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g'(x)\right)$. Showing students that this is not true, like the work to the right, can help them to see that this is not a correct formula. Keep an eye out for students making this mistake and correct them so that they will use the correct rule.

SUGGESTED SKILL 1.E

There are numerous rules and procedures that students will be required to learn throughout calculus. It is important that students be able to identify the rule that they need to use and apply it correctly. To help students remember rules, consider giving students formula quizzes where you ask them to write down selected derivative rules they have learned in the past.

Teaching Tip

While the proofs for the Product and Quotient Rules might seem challenging for students, they can teach students tools that mathematicians use to solve problems. In the proof of the Product Rule, we add zero $(-f(x+h)g(x)+f(x+h)g(x))$ to get the limit into a form that we can work with. The proof of the Quotient Rule uses a very similar step. Consider showing students the proof of the Product Rule and see if they can then prove the Quotient Rule on their own. Stress to students that adding zero and multiplying by one are both common procedures that mathematicians use to make problems easier to solve.

2.4 Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

OBJECTIVES *When you finish this section, you should be able to:*

1. **Differentiate the product of two functions (p. 194)**
2. **Differentiate the quotient of two functions (p. 196)**
3. **Find higher-order derivatives (p. 198)**
4. **Find the acceleration of an object in rectilinear motion (p. 200)**

In this section, we obtain formulas for differentiating products and quotients of functions. As it turns out, the formulas are not what we might expect. The derivative of the product of two functions is *not* the product of their derivatives, and the derivative of the quotient of two functions is *not* the quotient of their derivatives.

1 Differentiate the Product of Two Functions

Consider the two functions $f(x) = 2x$ and $g(x) = x^3$. Both are differentiable, and their derivatives are $f'(x) = 2$ and $g'(x) = 3x^2$. Form the product

$$F(x) = f(x)g(x) = 2x \cdot x^3 = 2x^4$$

Now find F' using the Constant Multiple Rule and the Simple Power Rule.

$$F'(x) = 2 \cdot 4x^3 = 8x^3$$

Notice that $f'(x)g'(x) = 2 \cdot 3x^2 = 6x^2$ is not equal to $F'(x) = \frac{d}{dx}[f(x)g(x)] = 8x^3$. We conclude that the derivative of a product of two functions is *not* the product of their derivatives.

To find the derivative of the product of two differentiable functions f and g, we let $F(x) = f(x)g(x)$ and use the definition of a derivative, namely,

$$F'(x) = \lim_{h\to 0} \frac{[f(x+h)g(x+h)] - [f(x)g(x)]}{h}$$

We can express F' in an equivalent form that contains the difference quotients for f and g, by subtracting and adding $f(x+h)g(x)$ to the numerator.

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + [f(x+h) - f(x)]g(x)}{h}$$ Group and factor.

$$= \left[\lim_{h\to 0} f(x+h)\right]\left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] + \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right]\left[\lim_{h\to 0} g(x)\right]$$ Use properties of limits.

$$= \left[\lim_{h\to 0} f(x+h)\right] g'(x) + f'(x)\left[\lim_{h\to 0} g(x)\right]$$ Definition of a derivative.

$$= f(x)g'(x) + f'(x)g(x)$$ $\lim_{h\to 0} g(x) = g(x)$ since h is not present. Since f is differentiable, it is continuous, so $\lim_{h\to 0} f(x+h) = f(x)$.

We have proved the following theorem.

TRM Alternate Examples Section 2.4

You can find the Alternate Examples for this section in PDF format in the Teacher's Resource Materials.

TRM AP® Calc Skill Builders Section 2.4

You can find the AP® Calc Skill Builders for this section in PDF format in the Teacher's Resource Materials.

THEOREM Product Rule

If f and g are differentiable functions and if $F(x)=f(x)g(x)$, then F is differentiable, and the derivative of the product F is

$$F'(x)=[f(x)g(x)]'=f(x)g'(x)+f'(x)g(x)$$

In Leibniz notation, the Product Rule has the form

$$\frac{d}{dx}F(x)=\frac{d}{dx}[f(x)g(x)]=f(x)\left[\frac{d}{dx}g(x)\right]+\left[\frac{d}{dx}f(x)\right]g(x)$$

IN WORDS The derivative of the product of two differentiable functions equals the first function times the derivative of the second function plus the derivative of the first function times the second function. That is,

$$(fg)'=f(g')+(f')g$$

EXAMPLE 1 Differentiating the Product of Two Functions

Find y' if $y=(1+x^2)e^x$.

Solution

The function y is the product of two functions: a polynomial, $f(x)=1+x^2$, and the exponential function, $g(x)=e^x$. By the Product Rule,

$$y'=\frac{d}{dx}[(1+x^2)e^x]\underset{\text{Product Rule}}{=}(1+x^2)\left[\frac{d}{dx}e^x\right]+\left[\frac{d}{dx}(1+x^2)\right]e^x=(1+x^2)e^x+2xe^x$$

At this point, we have found the derivative, but it is customary to simplify the answer. Then

$$y'\underset{\text{Factor out } e^x.}{=}(1+x^2+2x)e^x\underset{\text{Factor.}}{=}(x+1)^2e^x$$ ■

NOW WORK Problem 9 and AP® Practice Problem 4.

Do not use the Product Rule unnecessarily! When one of the factors is a constant, use the Constant Multiple Rule. For example, it is easier to work

$$\frac{d}{dx}[5(x^2+1)]=5\frac{d}{dx}(x^2+1)=5\cdot 2x=10x$$

than it is to work

$$\frac{d}{dx}[5(x^2+1)]=5\frac{d}{dx}(x^2+1)+\left[\frac{d}{dx}5\right](x^2+1)=5\cdot 2x+0\cdot(x^2+1)=10x$$

Also, it is easier to simplify $f(x)=x^2(4x-3)$ before finding the derivative. That is, it is easier to work

$$\frac{d}{dx}[x^2(4x-3)]=\frac{d}{dx}(4x^3-3x^2)=12x^2-6x$$

than it is to use the Product Rule

$$\frac{d}{dx}[x^2(4x-3)]=x^2\frac{d}{dx}(4x-3)+\left(\frac{d}{dx}x^2\right)(4x-3)=(x^2)(4)+(2x)(4x-3)$$
$$=4x^2+8x^2-6x=12x^2-6x$$

CALC CLIP

EXAMPLE 2 Differentiating a Product in Two Ways

Find the derivative of $F(v)=(5v^2-v+1)(v^3-1)$ in two ways:

(a) By using the Product Rule

(b) By multiplying the factors of the function before finding its derivative.

Teaching Tip

On the AP® Exam, students should feel comfortable working with functions that are presented as a function, table, or graph. Try to include examples throughout the year that allow students to practice working with functions in each of these representations. The alternate example below allows students to find the derivative of a product by using values from a table.

ALTERNATE EXAMPLE

Differentiating the Product of Two Functions

The functions f and g are differentiable for all real numbers. The table gives values of the functions and their first derivatives at selected values of x. Find the derivative of $y=f\circ g$ and then evaluate $y'(2)$.

Function	f	g	f'	g'
$x=2$	-1	3	2	-4

Solution

$y'=f\cdot g'+f\cdot g'$

$y'(2)=f(2)\cdot g'(2)+f'(2)\cdot g(2)$

$y'(2)=(-1)(-4)+(2)(3)$

$y'(2)=10$

AP® CALC SKILL BUILDER FOR EXAMPLE 2

Differentiating a Product in Two Ways

Find the derivative of $v(t)=(t^2-1)(t^2+1)$ in two ways:

(a) By using the Product Rule.

(b) By multiplying the factors of the function before finding its derivative.

Solution

(a) $v(t)=(t^2-1)(t^2+1)$

$v'(t)=(t^2-1)(2t)+(2t)(t^2+1)$

$v'(t)=2t^3-2t+2t^3+2t$

$v'(t)=4t^3$

(b) $v(t)=(t^2-1)(t^2+1)$

$v(t)=t^4-1$

$v'(t)=4t^3$

AP® CALC SKILL BUILDER FOR EXAMPLE 1

Differentiating the Product of Two Functions

Find y' if $y=\left(\frac{1}{2}x^2-2x\right)(1+2e^x)$.

Solution

By the Product Rule:

$$y'=\left(\frac{1}{2}x^2-2x\right)(2e^x)+(x-2)(1+2e^x)$$

$$=\left(\frac{1}{2}x^2-2x\right)(2e^x)+(x-2)+2e^x(x-2)$$

$$=\left(\frac{1}{2}x^2-2x+x-2\right)(2e^x)+(x-2)$$

$$=(x^2-2x-4)e^x+x-2$$

TRM Section 2.4: Worksheet 1

This worksheet includes 6 problems in which students are asked to find each derivative using the Product Rule. Then they are asked to verify the derivative, first by expanding, then by taking the derivative.

COMMON ERRORS & MISCONCEPTIONS

Students often interchange the two terms in the numerator of the Quotient Rule, which leads them to the opposite of the correct answer. Stress to students that subtraction is not commutative; thus, the order of terms in the numerator of the quotient rule matters.

Solution

(a) F is the product of the two functions $f(v)=5v^2-v+1$ and $g(v)=v^3-1$. Using the Product Rule, we get

$$\begin{aligned}F'(v)&=(5v^2-v+1)\left[\frac{d}{dv}(v^3-1)\right]+\left[\frac{d}{dv}(5v^2-v+1)\right](v^3-1)\\&=(5v^2-v+1)(3v^2)+(10v-1)(v^3-1)\\&=15v^4-3v^3+3v^2+10v^4-10v-v^3+1\\&=25v^4-4v^3+3v^2-10v+1\end{aligned}$$

(b) Here we multiply the factors of F before differentiating.

$$F(v)=(5v^2-v+1)(v^3-1)=5v^5-v^4+v^3-5v^2+v-1$$

Then

$$F'(v)=25v^4-4v^3+3v^2-10v+1$$ ■

Notice that the derivative is the same whether you differentiate and then simplify, or whether you multiply the factors and then differentiate. Use the approach that you find easier.

NOW WORK Problem 13.

2 Differentiate the Quotient of Two Functions

The derivative of the quotient of two functions is *not* equal to the quotient of their derivatives. Instead, the derivative of the quotient of two functions is found using the *Quotient Rule*.

THEOREM Quotient Rule

If two functions f and g are differentiable and if $F(x)=\dfrac{f(x)}{g(x)}$, $g(x)\neq 0$, then F is differentiable, and the derivative of the quotient F is

$$F'(x)=\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

In Leibniz notation, the Quotient Rule has the form

$$\frac{d}{dx}F(x)=\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\left[\frac{d}{dx}f(x)\right]g(x)-f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$$

IN WORDS The derivative of a quotient of two functions is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the denominator squared. That is,

$$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

Proof We use the definition of a derivative to find $F'(x)$.

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\underset{\uparrow\ F(x)=\frac{f(x)}{g(x)}}{=}\lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{h[g(x+h)g(x)]}$$

We write F' in an equivalent form that contains the difference quotients for f and g by subtracting and adding $f(x)g(x)$ to the numerator.

$$F'(x)=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h[g(x+h)g(x)]}$$

Now group and factor the numerator.

$$F'(x) = \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x) - f(x)[g(x+h) - g(x)]}{h[g(x+h)g(x)]}$$

$$= \lim_{h\to 0} \frac{\left[\frac{f(x+h)-f(x)}{h}\right] g(x) - f(x)\left[\frac{g(x+h)-g(x)}{h}\right]}{g(x+h)g(x)}$$

$$= \frac{\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\cdot \lim_{h\to 0} g(x) - \lim_{h\to 0} f(x)\cdot \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]}{\lim_{h\to 0} g(x+h)\cdot \lim_{h\to 0} g(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$ ■

RECALL Since g is differentiable, it is continuous; so, $\lim_{h\to 0} g(x+h) = g(x)$.

EXAMPLE 3 Differentiating the Quotient of Two Functions

Find y' if $y = \dfrac{x^2+1}{2x-3}$.

Solution

The function y is the quotient of $f(x) = x^2+1$ and $g(x) = 2x-3$. Using the Quotient Rule, we have

$$y' = \frac{d}{dx}\frac{x^2+1}{2x-3} = \frac{\left[\frac{d}{dx}(x^2+1)\right](2x-3) - (x^2+1)\left[\frac{d}{dx}(2x-3)\right]}{(2x-3)^2}$$

$$= \frac{(2x)(2x-3)-(x^2+1)(2)}{(2x-3)^2} = \frac{4x^2-6x-2x^2-2}{(2x-3)^2} = \frac{2x^2-6x-2}{(2x-3)^2}$$

provided $x \neq \frac{3}{2}$. ■

NOW WORK Problem 23 and AP® Practice Problems 1, 2, 3, 7 and 8.

COROLLARY Derivative of the Reciprocal of a Function

If a function g is differentiable, then

$$\frac{d}{dx}\frac{1}{g(x)} = -\frac{\frac{d}{dx}g(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2} \quad (1)$$

provided $g(x) \neq 0$.

IN WORDS The derivative of the reciprocal of a function is the negative of the derivative of the denominator divided by the square of the denominator. That is, $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$.

The proof of the corollary is left as an exercise. (See Problem 98.)

EXAMPLE 4 Differentiating the Reciprocal of a Function

(a) $$\frac{d}{dx}\frac{1}{x^2+x} = -\frac{\frac{d}{dx}(x^2+x)}{(x^2+x)^2} = -\frac{2x+1}{(x^2+x)^2}$$

Use (1).

(b) $$\frac{d}{dx}e^{-x} = \frac{d}{dx}\frac{1}{e^x} = -\frac{\frac{d}{dx}e^x}{(e^x)^2} = -\frac{e^x}{e^{2x}} = -\frac{1}{e^x} = -e^{-x}$$ ■

Use (1).

NOW WORK Problem 25.

AP® CALC SKILL BUILDER FOR EXAMPLE 3

Differentiating the Quotient of Two Functions

If $y = \dfrac{x^2+6x+2}{x+3}$, $x \neq -3$, find the instantaneous rate of change of y with respect to x at $x = 1$.

Solution

To find the instantaneous rate of change of y with respect to x, we first find the derivative of y with respect to x using the Quotient Rule.

$$y' = \frac{x^2+6x+2}{x+3}$$

$$= \frac{(2x+6)(x+3)-(x^2+6x+2)(1)}{(x+3)^2}$$

$$= \frac{2(x+3)^2-(x^2+6x+2)}{(x+3)^2}$$

$$= 2 - \frac{x^2+6x+2}{(x+3)^2}$$

At $x = 1$, the instantaneous rate of change of y with respect to x is

$$y' = 2 - \frac{1^2+6(1)+2}{(1+3)^2} = 2 - \frac{9}{16} = \frac{23}{16}$$

TRM Section 2.4: Worksheet 2

This worksheet includes 4 problems in which students find each derivative using the Quotient Rule. Then they are asked to verify the derivative, first by dividing, then by taking the derivative.

AP® CALC SKILL BUILDER FOR EXAMPLE 4

Differentiating the Reciprocal of a Function

Find the derivative

$$\frac{d}{dx}\frac{1}{x^3-3x^2+5}.$$

Solution

$$\frac{d}{dx}\left(\frac{1}{x^3-3x^2+5}\right) = -\frac{\frac{d}{dx}(x^3-3x^2+5)}{(x^3-3x^2+5)^2}$$

$$= -\frac{3x^2-6x}{(x^3-3x^2+5)^2}$$

AP® CALC SKILL BUILDER FOR EXAMPLE 3

Differentiating the Quotient of Two Functions

Find y' if $y = \dfrac{e^x}{3x+1}$.

Solution

$$y' = \frac{(e^x)(3x+1)-(e^x)(3)}{(3x+1)^2}$$

$$y' = \frac{3xe^x+e^x-3e^x}{(3x+1)^2}$$

$$y' = \frac{3xe^x-2e^x}{(3x+1)^2}$$

$$y' = \frac{e^x(3x-2)}{(3x+1)^2}$$

COMMON ERRORS & MISCONCEPTIONS

Students may be surprised that the derivative of a quotient is not simply the quotient of the individual derivatives. Consider $f(x) = \dfrac{(x^2-4)}{(x-2)}$. Ask the students to find the derivative of this function by using the Quotient Rule and then simplify their answer. Then ask them to find the derivative of the numerator and denominator separately. Compare the answers. The AP® Calc Skill Builder provided can also be factored and reduced. Do so, and then find the derivative. This will allow the students to see that the Quotient Rule provides the correct answer.

Notice that the derivative of the reciprocal of a function f is *not* the reciprocal of the derivative. That is,

$$\frac{d}{dx}\frac{1}{f(x)} \neq \frac{1}{f'(x)}$$

The rule for the derivative of the reciprocal of a function allows us to extend the Simple Power Rule to all integers. Here is the proof.

Suppose n is a negative integer and $x \neq 0$. Then $m = -n$ is a positive integer, and

$$\frac{d}{dx}x^n = \underset{\text{Use (1).}}{\frac{d}{dx}\frac{1}{x^m}} = -\frac{\frac{d}{dx}x^m}{(x^m)^2} \underset{\text{Simple Power Rule}}{=} -\frac{mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = -mx^{-m-1} \underset{\text{Substitute } n=-m.}{=} nx^{n-1} \blacksquare$$

NOTE In Section 2.3 we proved the Simple Power Rule, $\frac{d}{dx}x^n = nx^{n-1}$ where n is a positive integer. Here we have extended the Simple Power Rule from positive integers to all integers. In Chapter 3 we extend the result to include all real numbers.

THEOREM Power Rule

The derivative of $y = x^n$, where n is any integer, is

$$y' = \frac{d}{dx}x^n = nx^{n-1}$$

EXAMPLE 5 Differentiating Using the Power Rule

(a) $\frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$

(b) $\frac{d}{du}\frac{1}{u^2} = \frac{d}{du}u^{-2} = -2u^{-3} = -\frac{2}{u^3}$

(c) $\frac{d}{ds}\frac{4}{s^5} = 4\frac{d}{ds}s^{-5} = 4\cdot(-5)\,s^{-6} = -20s^{-6} = -\frac{20}{s^6}$ ■

NOW WORK Problem 31 and AP® Practice Problem 5.

EXAMPLE 6 Using the Power Rule in Electrical Engineering

Ohm's Law states that the current I running through a wire is inversely proportional to the resistance R in the wire and can be written as $I = \frac{V}{R}$, where V is the voltage. Find the rate of change of I with respect to R when $V = 12$ volts.

Solution

The rate of change of I with respect to R is the derivative $\frac{dI}{dR}$. We write Ohm's Law with $V = 12$ as $I = \frac{V}{R} = 12R^{-1}$ and use the Power Rule.

$$\frac{dI}{dR} = \frac{d}{dR}(12R^{-1}) = 12\cdot\frac{d}{dR}R^{-1} = 12(-1R^{-2}) = -\frac{12}{R^2}$$

The minus sign in $\frac{dI}{dR}$ indicates that the current I decreases as the resistance R in the wire increases. ■

NOW WORK Problem 91.

3 Find Higher-Order Derivatives

Since the derivative f' is a function, it makes sense to ask about the derivative of f'. The derivative (if there is one) of f' is also a function called the **second derivative** of f and denoted by f'', read "f double prime."

By continuing in this fashion, we can find the **third derivative** of f, the **fourth derivative** of f, and so on, provided that these derivatives exist. Collectively, these are called **higher-order derivatives**.

Teaching Tip

Have your students find the derivative of the function $y = \frac{1}{x}$ using the Quotient Rule. Then ask the students to rewrite the function as $y = x^{-1}$ and find the derivative using the Power Rule. Students should notice that both techniques gave them the same answer. As the students gain more experience in finding the derivative of functions, they will begin to see the pros and cons of the various techniques.

MATHEMATICAL PRACTICES TIP

Practice 1: Implementing Mathematical Processes

Throughout Chapters 2 and 3, students are learning a number of rules that they can use to find the derivative of a function without using the definition each time. Students will have an ever-increasing toolbox of techniques that they can use to find a derivative. As students learn more rules, ensure that you have conversations with them about when it is appropriate to use one rule as opposed to another. See the Teaching Tip below for an example of when it is better to rewrite a rational function to use the Power Rule instead of using the Quotient Rule.

Teaching Tip

Once students learn the Quotient Rule, many will want to use it to take the derivative of any rational expression. Remind students that if a function is in the form $\frac{c}{x^n}$ or $\frac{x^n}{c}$, where c and n are both real numbers, then the expressions can be rewritten so that the Power Rule can be used. For example, $\frac{c}{x^n} = c \circ x^{-n}$ and $\frac{x^n}{c} = \frac{1}{c} \circ x^n$. It is easier to use the Power Rule to take the derivative of functions in this form because there is less room for making a mistake than while using the Quotient Rule.

AP® EXAM TIP

Have the students memorize that the derivative of $y = \frac{1}{x}$ is $y' = -\frac{1}{x^2}$ and that the derivative of $y = \sqrt{x}$ is $y' = \frac{1}{2\sqrt{x}}$.

These derivatives show up regularly, so students can save time if they memorize these two rules.

AP® CALC SKILL BUILDER FOR EXAMPLE 5

Differentiating Using the Power Rule

$$\frac{d}{dx}(2\sqrt{x}) =$$

Solution

$$\frac{d}{dx}(2\sqrt{x}) = \frac{d}{dx}(2x^{1/2}) = x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$$

Or using the rule that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$,

$$\frac{d}{dx}(2\sqrt{x}) = 2\cdot\frac{d}{dx}\sqrt{x} = 2\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}.$$

Note: The full explanation of the Power Rule for rational exponents is in Chapter 3.

Leibniz notation also can be used for higher-order derivatives. Table 3 summarizes the notation for higher-order derivatives.

TABLE 3

	Prime Notation		Leibniz Notation	
First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}f(x)$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f(x)$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}f(x)$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}f(x)$
⋮				
nth Derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}f(x)$

EXAMPLE 7 Finding Higher-Order Derivatives of a Power Function

Find the second, third, and fourth derivatives of $y = 2x^3$.

Solution

Use the Power Rule and the Constant Multiple Rule to find each derivative. The first derivative is

$$y' = \frac{d}{dx}(2x^3) = 2 \cdot \frac{d}{dx}x^3 = 2 \cdot 3x^2 = 6x^2$$

The next three derivatives are

$$y'' = \frac{d^2}{dx^2}(2x^3) = \frac{d}{dx}(6x^2) = 6 \cdot \frac{d}{dx}x^2 = 6 \cdot 2x = 12x$$
$$y''' = \frac{d^3}{dx^3}(2x^3) = \frac{d}{dx}(12x) = 12$$
$$y^{(4)} = \frac{d^4}{dx^4}(2x^3) = \frac{d}{dx}12 = 0$$ ■

All derivatives of this function f of order 4 or more equal 0. This result can be generalized. For a power function f of degree n, where n is a positive integer,

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$
$$f''(x) = n(n-1)x^{n-2}$$
$$\vdots$$
$$f^{(n)}(x) = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

NOTE If $n > 1$ is an integer, the product $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ is often written $n!$ and is read, "n factorial." The **factorial symbol** ! means $0! = 1$, $1! = 1$, and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$, where $n > 1$.

The nth-order derivative of $f(x) = x^n$ is a constant, so all derivatives of order greater than n equal 0.

It follows from this discussion that the nth derivative of a polynomial of degree n is a constant and that all derivatives of order $n+1$ and higher equal 0.

NOW WORK Problem 41.

EXAMPLE 8 Finding Higher-Order Derivatives

Find the second and third derivatives of $y = (1 + x^2)e^x$.

Solution

In Example 1, we found that $y' = (1 + x^2)e^x + 2xe^x = (x^2 + 2x + 1)e^x$. To find y'', use the Product Rule with y'.

TRM Section 2.4: Worksheet 3

This worksheet includes 4 problems in which a student is asked to find the first four derivatives of each function.

AP® CALC SKILL BUILDER FOR EXAMPLE 7

Finding Higher-Order Derivatives of a Power Function

Find the first, second, third, and fourth derivatives of $y = -x^5 + 5x^4 + 2x$.

Solution

$$y = -x^5 + 5x^4 + 2x$$
$$y' = -5x^4 + 20x^3 + 2$$
$$y'' = -20x^3 + 60x^2$$
$$y''' = -60x^2 + 120x$$
$$y^{(4)} = -120x + 120$$

ALTERNATE EXAMPLE

Finding Higher-Order Derivatives

Find the second derivative of $y = \frac{e^x}{x}$.

Solution

$$y' = \frac{(e^x)(x) - (e^x)(1)}{x^2}$$
$$y' = \frac{e^x(x-1)}{x^2}$$
$$y'' = \frac{[(e^x)(x-1) + (e^x)(1)](x^2) - e^x(x-1)(2x)}{(x^2)^2}$$
$$y'' = \frac{[xe^x - e^x + e^x](x^2) - 2x^2e^x + 2xe^x}{x^4}$$
$$y'' = \frac{x^3e^x - 2x^2e^x + 2xe^x}{x^4}$$
$$y'' = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

Teaching Tip

Begin to build, in the students' minds, the relationship between position, velocity, and acceleration and the function, its first derivative, and its second derivative.

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

$$y'' = \frac{d}{dx}[(x^2+2x+1)e^x] = (x^2+2x+1)\left(\frac{d}{dx}e^x\right) + \left[\frac{d}{dx}(x^2+2x+1)\right]e^x$$

Product Rule

$$= (x^2+2x+1)e^x + (2x+2)e^x = (x^2+4x+3)e^x$$

$$y''' = \frac{d}{dx}[(x^2+4x+3)e^x] = (x^2+4x+3)\frac{d}{dx}e^x + \left[\frac{d}{dx}(x^2+4x+3)\right]e^x$$

Product Rule

$$= (x^2+4x+3)e^x + (2x+4)e^x = (x^2+6x+7)e^x$$ ■

NOW WORK Problem 45 and AP® Practice Problem 9.

4 Find the Acceleration of an Object in Rectilinear Motion

For an object in rectilinear motion whose signed distance s from the origin at time t is the position function $s = s(t)$, the derivative $s'(t)$ has a physical interpretation as the velocity of the object. The second derivative s'', which is the rate of change of the velocity, is called *acceleration*.

DEFINITION Acceleration

For an object in rectilinear motion, its signed distance s from the origin at time t is given by a position function $s = s(t)$. The first derivative $\frac{ds}{dt}$ is the velocity $v = v(t)$ of the object at time t.

The **acceleration** $a = a(t)$ of an object at time t is defined as the rate of change of velocity with respect to time. That is,

$$a = a(t) = \frac{dv}{dt} = \frac{d}{dt}v = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

IN WORDS Acceleration is the second derivative of a position function with respect to time.

EXAMPLE 9 Analyzing Vertical Motion

A ball is propelled vertically upward from the ground with an initial velocity of 29.4 m/s. The height s (in meters) of the ball above the ground is approximately $s = s(t) = -4.9t^2 + 29.4t$, where t is the number of seconds that elapse from the moment the ball is released.

(a) What is the velocity of the ball at time t? What is its velocity at $t = 1$ s?

(b) When will the ball reach its maximum height?

(c) What is the maximum height the ball reaches?

(d) What is the acceleration of the ball at any time t?

(e) How long is the ball in the air?

(f) What is the velocity of the ball upon impact with the ground? What is its speed?

(g) What is the total distance traveled by the ball?

Solution

(a) Since $s = s(t) = -4.9t^2 + 29.4t$, then

$$v = v(t) = \frac{ds}{dt} = -9.8t + 29.4$$

$$v(1) = -9.8 + 29.4 = 19.6$$

At $t = 1$ s, the velocity of the ball is 19.6 m/s.

AP® CALC SKILL BUILDER FOR EXAMPLE 9

Analyzing Rectilinear Motion

A particle moves along the x-axis so that at any time t the position of the particle is given by $x(t) = 2te^{-t}$. Find all values of t for which the particle is at rest. At what time is the acceleration of the particle 0?

Solution

The particle will be at rest when its velocity is 0.

Rewrite the function as $x(t) = \frac{2t}{e^t}$ and find the derivative:

$$v(t) = \frac{(2)(e^t) - (2t)(e^t)}{e^{2t}}$$

$$0 = \frac{(2)(e^t) - (2t)(e^t)}{e^{2t}}$$

$$0 = \frac{2e^t(1-t)}{e^{2t}}$$

$$0 = 2e^t - t(1-t)$$

$$t = 1$$

The particle will be at rest when $t = 1$.

To find where $a(t) = 0$, compute the derivative of $v(t) = \frac{2-2t}{e^t}$ using the Quotient Rule:

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(\frac{2-2t}{e^t}\right)$$

$$= \frac{-2e^t - (2-2t)e^t}{e^{2t}} = \frac{2(t-2)}{e^t}$$

Now set the expression equal to 0 and solve.

$$\frac{2(t-2)}{e^t} = 0$$

$$2(t-2) = 0$$

$$t = 2$$

The acceleration of the particle is 0 at $t = 2$.

AP® EXAM TIP

On the AP® Exam, position functions are often referred to as $x(t)$ instead of $s(t)$. When doing examples in class, it is a good idea to use both x and s for position functions so that students are familiar with using both.

(b) The ball reaches its maximum height when $v(t) = 0$.

$$v(t) = -9.8t + 29.4 = 0$$
$$9.8t = 29.4$$
$$t = 3$$

The ball reaches its maximum height after 3 s.

(c) The maximum height is

$$s = s(3) = -4.9 \cdot 3^2 + 29.4 \cdot 3 = 44.1$$

The maximum height of the ball is 44.1 m.

(d) The acceleration of the ball at any time t is

$$a = a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(-9.8t + 29.4) = -9.8 \text{ m/s}^2$$

NOTE *Speed* and *velocity* are not the same. Speed measures how fast an object is moving and is defined as the absolute value of its velocity. Velocity measures both the speed and the direction of an object and may be a positive number or a negative number or zero.

(e) There are two ways to answer the question "How long is the ball in the air?" *First way:* Since it takes 3 s for the ball to reach its maximum height, it follows that it will take another 3 s to reach the ground, for a total time of 6 s in the air. *Second way:* When the ball reaches the ground, $s = s(t) = 0$. Solve for t:

$$s(t) = -4.9t^2 + 29.4t = 0$$
$$t(-4.9t + 29.4) = 0$$
$$t = 0 \quad \text{or} \quad t = \frac{29.4}{4.9} = 6$$

The ball is at ground level at $t = 0$ and at $t = 6$, so the ball is in the air for 6 s.

(f) Upon impact with the ground, $t = 6$ s. So the velocity is

$$v(6) = (-9.8)(6) + 29.4 = -29.4$$

Upon impact the direction of the ball is downward, and its speed is 29.4 m/s.

(g) The total distance traveled by the ball is

$$s(3) + s(3) = 2\,s(3) = 2(44.1) = 88.2 \text{ m}$$ ■

See Figure 29 for an illustration.

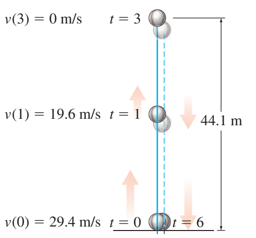


Figure 29

NOW WORK Problem 83 and AP® Practice Problem 6.

In Example 9, the acceleration of the ball is constant. In fact, acceleration is the same for all falling objects at the same location, provided air resistance is not taken into account. In the sixteenth century, Galileo (1564–1642) discovered this by experimentation.* He also found that all falling bodies obey the law, stating that the distance s they fall when dropped is proportional to the square of the time t it takes to fall that distance, and that the constant of proportionality c is the same for all objects. That is,

$$s = -ct^2$$

NOTE The Earth is not perfectly round; it bulges slightly at the equator, and its mass is not distributed uniformly. As a result, the acceleration of a freely falling body varies slightly.

*In a famous legend, Galileo dropped a feather and a rock from the top of the Leaning Tower of Pisa, to show that the acceleration due to gravity is constant. He expected them to fall together, but he failed to account for air resistance that slowed the feather. In July 1971, *Apollo* 15 astronaut David Scott repeated the experiment on the Moon, where there is no air resistance. He dropped a hammer and a falcon feather from his shoulder height. Both hit the Moon's surface at the same time. A video of this experiment may be found at the NASA website.

Teaching Tip

Particle-motion problems commonly appear on the AP® Calculus Exam. To preview coming attractions, some teachers mention that this is 1-dimensional motion in which the particle can only move along a horizontal or vertical axis. Calculus BC students will study particle motion in two dimensions in Chapter 9 using vectors, so position, velocity, and acceleration can vary in both the x and y directions. In the real world, these quantities vary in all three directions, x, y, and z. Students might study these topics in college in a Multivariable Calculus course.

MUST-DO PROBLEMS FOR EXAM READINESS

AB: 9, 13, 19, 23, 25, 31, 37, 41, 45, 57, 67, 81, 83, 91, and all AP® Practice Problems

BC: 9, 13, 23, 25, 31, 41, 45, 47, 69, 77, 81, 83, 91, and all AP® Practice Problems

TRM Full Solutions to Section 2.4 Problems and AP® Practice Problems

Answers to Section 2.4 Problems

1. False.
2. $f(x)g'(x)+f'(x)g(x)$
3. False.
4. $\dfrac{\left[\dfrac{d}{dx}f(x)\right]g(x)-f(x)\left[\dfrac{d}{dx}g(x)\right]}{[g(x)]^2}$
5. True.
6. $-\dfrac{\dfrac{d}{dx}g(x)}{[g(x)]^2}$
7. 0
8. $\dfrac{d^2s}{dt^2}$
9. $e^x(x+1)$
10. $x(x+2)e^x$
11. $5x^4-2x$
12. $5x^4+20x^3$
13. $18x^2+6x-10$
14. $24x+7$
15. $16t^7-24t^5+10t^4-4t^3+4t-1$
16. $6u^5-5u^4-4u^3+9u^2-10u-1$
17. $x^3e^x+3x^2e^x+e^x+3x^2$
18. $x^2e^x+2xe^x+e^x+3x^2+1$
19. $\dfrac{2}{(s+1)^2}$
20. $-\dfrac{1}{2z^2}$
21. $-\dfrac{4}{(1+2u)^2}$
22. $-\dfrac{4w}{(1+w^2)^2}$
23. $\dfrac{2(6x^2+16x+3)}{(3x+4)^2}$
24. $\dfrac{-6x^4-9x^2+4x}{(2x^2+1)^2}$
25. $-\dfrac{3w^2}{(w^3-1)^2}$
26. $-\dfrac{2v+5}{(v^2+5v-1)^2}$
27. $-\dfrac{3}{t^4}$
28. $-\dfrac{4}{u^5}$
29. $\dfrac{4}{e^x}$
30. $-\dfrac{3}{4e^x}$
31. $-\dfrac{40}{x^5}-\dfrac{6}{x^3}$
32. $-\dfrac{10}{x^6}+\dfrac{9}{x^4}$

The velocity v of the falling object is

$$v=\frac{ds}{dt}=\frac{d}{dt}(-ct^2)=-2ct$$

and its acceleration a is

$$a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=-2c$$

which is a constant. Usually, we denote the constant $2c$ by g so $c=\frac{1}{2}g$. Then

$$a=-g \qquad v=-gt \qquad s=-\frac{1}{2}gt^2$$

The number g is called the **acceleration due to gravity**. For our planet, g is approximately 32 ft/s^2, or 9.8 m/s^2. On the planet Jupiter, $g\approx 26.0$ m/s^2, and on our moon, $g\approx 1.60$ m/s^2.

2.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative of a product is the product of the derivatives.
2. If $F(x)=f(x)g(x)$, then $F'(x)=$ ______.
3. *True or False* $\dfrac{d}{dx}x^n=nx^{n+1}$, for any integer n.
4. If f and $g\neq 0$ are two differentiable functions, then $\dfrac{d}{dx}\dfrac{f(x)}{g(x)}=$ ______.
5. *True or False* $f(x)=\dfrac{e^x}{x^2}$ can be differentiated using the Quotient Rule or by writing $f(x)=\dfrac{e^x}{x^2}=x^{-2}e^x$ and using the Product Rule.
6. If $g\neq 0$ is a differentiable function, then $\dfrac{d}{dx}\dfrac{1}{g(x)}=$ ______.
7. If $f(x)=x$, then $f''(x)=$ ______.
8. When an object in rectilinear motion is modeled by the position function $s=s(t)$, then the acceleration a of the object at time t is given by $a=a(t)=$ ______.

Skill Building

In Problems 9–40, find the derivative of each function.

PAGE 195 **9.** $f(x)=xe^x$
10. $f(x)=x^2e^x$
11. $f(x)=x^2(x^3-1)$
12. $f(x)=x^4(x+5)$
PAGE 196 **13.** $f(x)=(3x^2-5)(2x+1)$
14. $f(x)=(3x-2)(4x+5)$
15. $s(t)=(2t^5-t)(t^3-2t+1)$
16. $F(u)=(u^4-3u^2+1)(u^2-u+2)$
17. $f(x)=(x^3+1)(e^x+1)$
18. $f(x)=(x^2+1)(e^x+x)$
19. $g(s)=\dfrac{2s}{s+1}$
20. $F(z)=\dfrac{z+1}{2z}$
21. $G(u)=\dfrac{1-2u}{1+2u}$
22. $f(w)=\dfrac{1-w^2}{1+w^2}$
PAGE 197 **23.** $f(x)=\dfrac{4x^2-2}{3x+4}$
24. $f(x)=\dfrac{-3x^3-1}{2x^2+1}$
PAGE 197 **25.** $f(w)=\dfrac{1}{w^3-1}$
26. $g(v)=\dfrac{1}{v^2+5v-1}$
27. $s(t)=t^{-3}$
28. $G(u)=u^{-4}$
29. $f(x)=-\dfrac{4}{e^x}$
30. $f(x)=\dfrac{3}{4e^x}$
PAGE 198 **31.** $f(x)=\dfrac{10}{x^4}+\dfrac{3}{x^2}$
32. $f(x)=\dfrac{2}{x^5}-\dfrac{3}{x^3}$
33. $f(x)=3x^3-\dfrac{1}{3x^2}$
34. $f(x)=x^5-\dfrac{5}{x^5}$
35. $s(t)=\dfrac{1}{t}-\dfrac{1}{t^2}+\dfrac{1}{t^3}$
36. $s(t)=\dfrac{1}{t}+\dfrac{1}{t^2}+\dfrac{1}{t^3}$
37. $f(x)=\dfrac{e^x}{x^2}$
38. $f(x)=\dfrac{x^2}{e^x}$
39. $f(x)=\dfrac{x^2+1}{xe^x}$
40. $f(x)=\dfrac{xe^x}{x^2-x}$

In Problems 41–54, find f' and f'' for each function.

PAGE 199 **41.** $f(x)=3x^2+x-2$
42. $f(x)=-5x^2-3x$
43. $f(x)=e^x-3$
44. $f(x)=x-e^x$
PAGE 200 **45.** $f(x)=(x+5)e^x$
46. $f(x)=3x^4e^x$
47. $f(x)=(2x+1)(x^3+5)$
48. $f(x)=(3x-5)(x^2-2)$
49. $f(x)=x+\dfrac{1}{x}$
50. $f(x)=x-\dfrac{1}{x}$
51. $f(t)=\dfrac{t^2-1}{t}$
52. $f(u)=\dfrac{u+1}{u}$
53. $f(x)=\dfrac{e^x+x}{x}$
54. $f(x)=\dfrac{e^x}{x}$
55. Find y' and y'' for **(a)** $y=\dfrac{1}{x}$ and **(b)** $y=\dfrac{2x-5}{x}$.
56. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for **(a)** $y=\dfrac{5}{x^2}$ and **(b)** $y=\dfrac{2-3x}{x}$.

33. $9x^2+\dfrac{2}{3x^3}$
34. $5x^4+\dfrac{25}{x^6}$
35. $-\dfrac{1}{t^2}+\dfrac{2}{t^3}-\dfrac{3}{t^4}$
36. $-\dfrac{1}{t^2}-\dfrac{2}{t^3}-\dfrac{3}{t^4}$
37. $\dfrac{(x-2)e^x}{x^3}$
38. $-\dfrac{x(x-2)}{e^x}$
39. $-\dfrac{x^3-x^2+x+1}{x^2e^x}$
40. $\dfrac{(x-2)e^x}{(x-1)^2}$
41. $f'(x)=6x+1, f''(x)=6$
42. $f'(x)=-10x-3, f''(x)=-10$
43. $f'(x)=e^x, f''(x)=e^x$
44. $f'(x)=1-e^x, f''(x)=-e^x$
45. $f'(x)=e^x(x+6), f''(x)=e^x(x+7)$
46. $f'(x)=3x^3(x+4)e^x$, $f''(x)=3x^2(x+6)(x+2)e^x$
47. $f'(x)=8x^3+3x^2+10$, $f''(x)=24x^2+6x$
48. $f'(x)=9x^2-10x-6, f''(x)=18x-10$
49. $f'(x)=1-\dfrac{1}{x^2}, f''(x)=\dfrac{2}{x^3}$
50. $f'(x)=1+\dfrac{1}{x^2}, f''(x)=-\dfrac{2}{x^3}$
51. $f'(t)=1+\dfrac{1}{t^2}, f''(t)=-\dfrac{2}{t^3}$

Answers continue on p. 203

Rectilinear Motion *In Problems 57–60, find the velocity $v = v(t)$ and acceleration $a = a(t)$ of an object in rectilinear motion whose signed distance s from the origin at time t is modeled by the position function $s = s(t)$.*

57. $s(t)=16t^2+20t$ **58.** $s(t)=16t^2+10t+1$

59. $s(t)=4.9t^2+4t+4$ **60.** $s(t)=4.9t^2+5t$

In Problems 61–68, find the indicated derivative.

61. $f^{(4)}(x)$ if $f(x)=x^3-3x^2+2x-5$

62. $f^{(5)}(x)$ if $f(x)=4x^3+x^2-1$

63. $\dfrac{d^8}{dt^8}\left(\dfrac{1}{8}t^8-\dfrac{1}{7}t^7+t^5-t^3\right)$ **64.** $\dfrac{d^6}{dt^6}(t^6+5t^5-2t+4)$

65. $\dfrac{d^7}{du^7}(e^u+u^2)$ **66.** $\dfrac{d^{10}}{du^{10}}(2e^u)$

67. $\dfrac{d^5}{dx^5}(-e^x)$ **68.** $\dfrac{d^8}{dx^8}(12x-e^x)$

In Problems 69–72:

(a) Find the slope of the tangent line for each function f at the given point.
(b) Find an equation of the tangent line to the graph of each function f at the given point.
(c) Find the points, if any, where the graph of the function has a horizontal tangent line.
(d) Graph each function, the tangent line found in (b), and any tangent lines found in (c) on the same set of axes.

69. $f(x)=\dfrac{x^2}{x-1}$ at $\left(-1,-\dfrac{1}{2}\right)$ **70.** $f(x)=\dfrac{x}{x+1}$ at $(0,0)$

71. $f(x)=\dfrac{x^3}{x+1}$ at $\left(1,\dfrac{1}{2}\right)$ **72.** $f(x)=\dfrac{x^2+1}{x}$ at $\left(2,\dfrac{5}{2}\right)$

In Problems 73–80:

(a) Find the points, if any, at which the graph of each function f has a horizontal tangent line.
(b) Find an equation for each horizontal tangent line.
(c) Solve the inequality $f'(x)>0$.
(d) Solve the inequality $f'(x)<0$.
(e) Graph f and any horizontal lines found in (b) on the same set of axes.
(f) Describe the graph of f for the results obtained in (c) and (d).

73. $f(x)=(x+1)(x^2-x-11)$ **74.** $f(x)=(3x^2-2)(2x+1)$

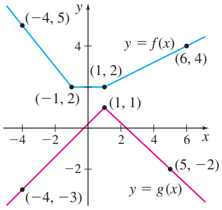
75. $f(x)=\dfrac{x^2}{x+1}$ **76.** $f(x)=\dfrac{x^2+1}{x}$

77. $f(x)=xe^x$ **78.** $f(x)=x^2e^x$

79. $f(x)=\dfrac{x^2-3}{e^x}$ **80.** $f(x)=\dfrac{e^x}{x^2+1}$

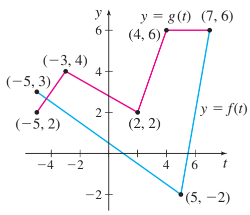
In Problems 81 and 82, use the graphs to determine each derivative.

81. Let $u(x)=f(x)\cdot g(x)$ and $v(x)=\dfrac{g(x)}{f(x)}$.



(a) $u'(0)$ (b) $u'(4)$
(c) $v'(-2)$ (d) $v'(6)$
(e) $\dfrac{d}{dx}\dfrac{1}{f(x)}$ at $x=-2$ (f) $\dfrac{d}{dx}\dfrac{1}{g(x)}$ at $x=4$

82. Let $F(t)=f(t)\cdot g(t)$ and $G(t)=\dfrac{f(t)}{g(t)}$.



(a) $F'(0)$ (b) $F'(3)$
(c) $F'(-4)$ (d) $G'(-2)$
(e) $G'(-1)$ (f) $\dfrac{d}{dt}\dfrac{1}{f(t)}$ at $t=3$

Applications and Extensions

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83. Vertical Motion An object is propelled vertically upward from the ground with an initial velocity of 39.2 m/s. The distance s (in meters) of the object from the ground after t seconds is given by the position function $s=s(t)=-4.9t^2+39.2t$.

(a) What is the velocity of the object at time t?
(b) When will the object reach its maximum height?
(c) What is the maximum height?
(d) What is the acceleration of the object at any time t?
(e) How long is the object in the air?
(f) What is the velocity of the object upon impact with the ground? What is its speed?
(g) What is the total distance traveled by the object?

52. $f'(u)=-\dfrac{1}{u^2}, f''(u)=\dfrac{2}{u^3}$

53. $f'(x)=e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$, $f''(x)=e^x\left(\dfrac{2}{x^3}-\dfrac{2}{x^2}+\dfrac{1}{x}\right)$

54. $f'(x)=\dfrac{(x-1)e^x}{x^2}$, $f''(x)=\dfrac{e^x(x^2-2x+2)}{x^3}$

55. (a) $y'=-\dfrac{1}{x^2}, y''=\dfrac{2}{x^3}$ (b) $y'=\dfrac{5}{x^2}, y''=-\dfrac{10}{x^3}$

56. (a) $\dfrac{dy}{dx}=-\dfrac{10}{x^3}, \dfrac{d^2y}{dx^2}=\dfrac{30}{x^4}$ (b) $\dfrac{dy}{dx}=-\dfrac{2}{x^2}, \dfrac{d^2y}{dx^2}=\dfrac{4}{x^3}$

57. $v(t)=32t+20, a(t)=32$

58. $v(t)=32t+10, a(t)=32$

59. $v(t)=9.8t+4, a(t)=9.8$

60. $v(t)=9.8t+5, a(t)=9.8$

61. $f^{(4)}(x)=0$

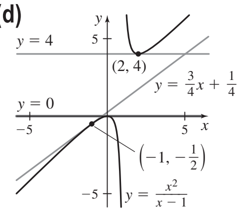
62. $f^{(5)}(x)=0$

63. 5040 **64.** 720

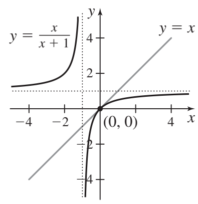
65. e^u **66.** $2e^u$

67. $-e^x$ **68.** $-e^x$

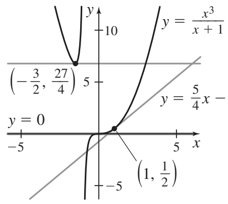
69. (a) $\dfrac{3}{4}$ (b) $y=\dfrac{3}{4}x+\dfrac{1}{4}$ (c) (0, 0), (2, 4) (d)



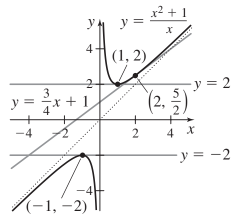
70. (a) 1 (b) $y=x$ (c) None. (d)



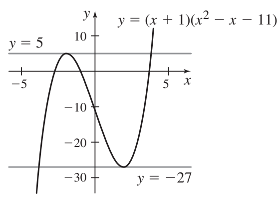
71. (a) $\dfrac{5}{4}$ (b) $y=\dfrac{5}{4}x-\dfrac{3}{4}$ (c) $(0,0), \left(-\dfrac{3}{2},\dfrac{27}{4}\right)$ (d)



72. (a) $\dfrac{3}{4}$ (b) $y=\dfrac{3}{4}x+1$ (c) $(1, 2), (-1, -2)$ (d)



73. (a) $(-2, 5), (2, -27)$ (b) $y=5, y=-27$ (c) $x<-2$ or $x>2$ (d) $-2<x<2$ (e)

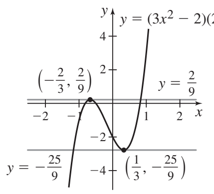


(f) f increasing on (−°, −2) and (2, ∞) where $f'(x)>0$, decreasing on (−2, 2), where $f'(x)<0$.

74. (a) $\left(-\dfrac{2}{3},\dfrac{2}{9}\right)\left(\dfrac{1}{3},-\dfrac{25}{9}\right)$ (b) $y=\dfrac{2}{9}, y=-\dfrac{25}{9}$ (c) $x<-\dfrac{2}{3}$ or $x>\dfrac{1}{3}$ (d) $-\dfrac{2}{3}<x<\dfrac{1}{3}$

Answers continue on p. 204

(e)



(f) f increasing on $\left(-°, -\frac{2}{3}\right)$ and $\left(\frac{1}{3}, \infty\right)$, where $f'(x) > 0$, decreasing on $\left(-\frac{2}{3}, \frac{1}{3}\right)$, where $f'(x) < 0$.

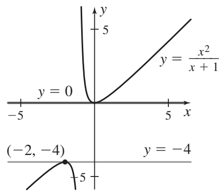
75. (a) $(0, 0)$, $(-2, -4)$

(b) $y = 0$, $y = -4$

(c) $x < -2$ or $x > 0$

(d) $-2 < x < -1$ or $-1 < x < 0$

(e)



(f) f increasing on $(-°, -2)$ and $(0, \infty)$ where $f'(x) > 0$, decreasing on $(-2, -1)$ and $(-1, 0)$, where $f'(x) < 0$.

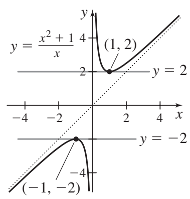
76. (a) $(1, 2)$, $(-1, -2)$

(b) $y = 2$, $y = -2$

(c) $x < -1$ or $x > 1$

(d) $-1 < x < 0$ or $0 < x < 1$

(e)

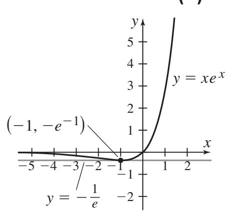


(f) f increasing on $(-°, -1)$ and $(1, \infty)$, where $f'(x) > 0$, decreasing on $(-1, 0)$ and $(0, 1)$, where $f'(x) < 0$.

77. (a) $\left(-1, -\frac{1}{e}\right)$ **(b)** $y = -\frac{1}{e}$

(c) $x > -1$ **(d)** $x < -1$

(e)



(f) f increasing on $(-1, °)$, where $f'(x) > 0$, decreasing on $(-°, -1)$, where $f'(x) < 0$.

84. Vertical Motion A ball is thrown vertically upward from a height of 6 ft with an initial velocity of 80 ft/s. The distance s (in feet) of the ball from the ground after t seconds is given by the position function $s = s(t) = 6 + 80t - 16t^2$.

(a) What is the velocity of the ball after 2 s?
(b) When will the ball reach its maximum height?
(c) What is the maximum height the ball reaches?
(d) What is the acceleration of the ball at any time t?
(e) How long is the ball in the air?
(f) What is the velocity of the ball upon impact with the ground? What is its speed?
(g) What is the total distance traveled by the ball?

85. Environmental Cost The cost C, in thousands of dollars, for the removal of a pollutant from a certain lake is given by the function $C(x) = \frac{5x}{110 - x}$, where x is the percent of pollutant removed.

(a) What is the domain of C?
(b) Graph C.
(c) What is the cost to remove 80% of the pollutant?
(d) Find $C'(x)$, the rate of change of the cost C with respect to the amount of pollutant removed.
(e) Find the rate of change of the cost for removing 40%, 60%, 80%, and 90% of the pollutant.
(f) Interpret the answers found in (e).

86. Investing in Fine Art The value V of a painting t years after it is purchased is modeled by the function

$$V(t) = \frac{100t^2 + 50}{t} + 400 \quad 1 \le t \le 5$$

(a) Find the rate of change in the value V with respect to time.
(b) What is the rate of change in value after 2 years?
(c) What is the rate of change in value after 3 years?
(d) Interpret the answers in (b) and (c).

87. Drug Concentration The concentration of a drug in a patient's blood t hours after injection is given by the function $f(t) = \frac{0.4t}{2t^2 + 1}$ (in milligrams per liter).

(a) Find the rate of change of the concentration with respect to time.
(b) What is the rate of change of the concentration after 10 min? After 30 min? After 1 hour?
(c) Interpret the answers found in (b).
(d) Graph f for the first 5 hours after administering the drug.
(e) From the graph, approximate the time (in minutes) at which the concentration of the drug is highest. What is the highest concentration of the drug in the patient's blood?

88. Population Growth A population of 1000 bacteria is introduced into a culture and grows in number according to the formula $P(t) = 1000\left(1 + \frac{4t}{100 + t^2}\right)$, where t is measured in hours.

(a) Find the rate of change in population with respect to time.
(b) What is the rate of change in population at $t = 1$, $t = 2$, $t = 3$, and $t = 4$?
(c) Interpret the answers found in (b).
(d) Graph $P = P(t)$, $0 \le t \le 20$.
(e) From the graph, approximate the time (in hours) when the population is the greatest. What is the maximum population of the bacteria in the culture?

89. Economics The price-demand function for a popular e-book is given by $D(p) = \frac{100,000}{p^2 + 10p + 50}$, $4 \le p \le 20$, where $D = D(p)$ is the quantity demanded at the price p dollars.

(a) Find $D'(p)$, the rate of change of demand with respect to price.
(b) Find $D'(5)$, $D'(10)$, and $D'(15)$.
(c) Interpret the results found in (b).

90. Intensity of Light The intensity of illumination I on a surface is inversely proportional to the square of the distance r from the surface to the source of light. If the intensity is 1000 units when the distance is 1 m from the light, find the rate of change of the intensity with respect to the distance when the source is 10 meters from the surface.

PAGE 198 **91. Ideal Gas Law** The Ideal Gas Law, used in chemistry and thermodynamics, relates the pressure p, the volume V, and the absolute temperature T (in Kelvin) of a gas, using the equation $pV = nRT$, where n is the amount of gas (in moles) and $R = 8.31$ is the ideal gas constant. In an experiment, a spherical gas container of radius r meters is placed in a pressure chamber and is slowly compressed while keeping its temperature at 273 K.

(a) Find the rate of change of the pressure p with respect to the radius r of the chamber.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) If the sphere contains 1.0 mol of gas, find the rate of change of the pressure when $r = \frac{1}{4}$ m.
Note: The metric unit of pressure is the pascal, Pa.

92. Body Density The density ρ of an object is its mass m divided by its volume V; that is, $\rho = \frac{m}{V}$. If a person dives below the surface of the ocean, the water pressure on the diver will steadily increase, compressing the diver and therefore increasing body density. Suppose the diver is modeled as a sphere of radius r.

(a) Find the rate of change of the diver's body density with respect to the radius r of the sphere.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) Find the rate of change of the diver's body density when the radius is 45 cm and the mass is 80,000 g (80 kg).

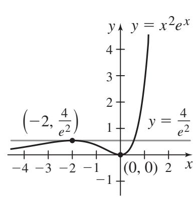
Jerk and Snap *Problems 93–96 use the following discussion:* Suppose that an object is moving in rectilinear motion so that its signed distance s from the origin at time t is given by the position function $s = s(t)$. The velocity $v = v(t)$ of the object at time t is the rate of change of s with respect to time, namely, $v = v(t) = \frac{ds}{dt}$. The acceleration $a = a(t)$

78. (a) $(0, 0)$ and $\left(-2, \frac{4}{e^2}\right)$

(b) $y = 0$, $y = \frac{4}{e^2}$ **(c)** $x < -2$ or $x > 0$

(d) $-2 < x < 0$

(e)



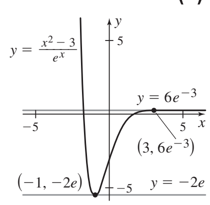
(f) f increasing on $(-°, -2)$ and $(0, \infty)$, where $f'(x) > 0$, decreasing on $(-2, 0)$ where $f'(x) < 0$.

79. (a) $(-1, -2e)$, $\left(3, \frac{6}{e^3}\right)$

(b) $y = -2e$, $y = \frac{6}{e^3}$

(c) $-1 < x < 3$ **(d)** $x < -1$ or $x > 3$

(e)



(f) f increasing on $(-1, 3)$, where $f'(x) > 0$, decreasing on $(-°, -1)$ and $(3, \infty)$, where $f'(x) < 0$.

Answers continue on p. 205

of the object at time t is the rate of change of the velocity with respect to time,

$$a = a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

There are also physical interpretations of the third derivative and the fourth derivative of $s = s(t)$. The **jerk** $J = J(t)$ of the object at time t is the rate of change of the acceleration a with respect to time; that is,

$$J = J(t) = \frac{da}{dt} = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

The **snap** $S = S(t)$ of the object at time t is the rate of change of the jerk J with respect to time; that is,

$$S = S(t) = \frac{dJ}{dt} = \frac{d^2a}{dt^2} = \frac{d^3v}{dt^3} = \frac{d^4s}{dt^4}$$

Engineers take jerk into consideration when designing elevators, aircraft, and cars. In these cases, they try to minimize jerk, making for a smooth ride. But when designing thrill rides, such as roller coasters, the jerk is increased, making for an exciting experience.

93. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = t^3 - t + 1$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at time t.
(b) When is the velocity of the object 0 m/s?
(c) Find the acceleration of the object at $t = 2$ and at $t = 5$.
(d) Does the jerk of the object ever equal 0 m/s^3?
(e) How would you interpret the snap for this object in rectilinear motion?

94. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = \frac{1}{6}t^4 - t^2 + \frac{1}{2}t + 4$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at any time t.
(b) Find the velocity of the object at $t = 0$ and at $t = 3$.
(c) Find the acceleration of the object at $t = 0$. Interpret your answer.
(d) Is the jerk of the object constant? In your own words, explain what the jerk says about the acceleration of the object.
(e) How would you interpret the snap for this object in rectilinear motion?

95. Elevator Ride Quality The ride quality of an elevator depends on several factors, two of which are acceleration and jerk. In a study of 367 persons riding in a 1600-kg elevator that moves at an average speed of 4 m/s, the majority of riders were comfortable in an elevator with vertical motion given by

$$s(t) = 4t + 0.8t^2 + 0.333t^3$$

(a) Find the acceleration that the riders found acceptable.
(b) Find the jerk that the riders found acceptable.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

96. Elevator Ride Quality In a hospital, the effects of high acceleration or jerk may be harmful to patients, so the acceleration and jerk need to be lower than in standard elevators. It has been determined that a 1600-kg elevator that is installed in a hospital and that moves at an average speed of 4 m/s should have vertical motion

$$s(t) = 4t + 0.55t^2 + 0.1167t^3$$

(a) Find the acceleration of a hospital elevator.
(b) Find the jerk of a hospital elevator.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

97. Current Density in a Wire The current density J in a wire is a measure of how much an electrical current is compressed as it flows through a wire and is modeled by the function $J(A) = \frac{I}{A}$, where I is the current (in amperes) and A is the cross-sectional area of the wire. In practice, current density, rather than merely current, is often important. For example, superconductors lose their superconductivity if the current density is too high.

(a) As current flows through a wire, it heats the wire, causing it to expand in area A. If a constant current is maintained in a cylindrical wire, find the rate of change of the current density J with respect to the radius r of the wire.
(b) Interpret the sign of the answer found in (a).
(c) Find the rate of change of current density with respect to the radius r when a current of 2.5 amps flows through a wire of radius $r = 0.50$ mm.

98. Derivative of a Reciprocal, Function Prove that if a function g is differentiable, then $\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{[g(x)]^2}$, provided $g(x) \neq 0$.

99. Extended Product Rule Show that if f, g, and h are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

From this, deduce that

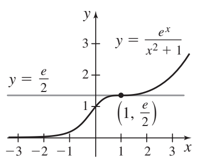
$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

In Problems 100–105, use the Extended Product Rule (Problem 99) to find y'.

100. $y = (x^2+1)(x-1)(x+5)$
101. $y = (x-1)(x^2+5)(x^3-1)$
102. $y = (x^4+1)^3$
103. $y = (x^3+1)^3$
104. $y = (3x+1)\left(1+\frac{1}{x}\right)(x^{-5}+1)$
105. $y = \left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\left(1-\frac{1}{x^3}\right)$

106. (Further) Extended Product Rule Write a formula for the derivative of the product of four differentiable functions. That is, find a formula for $\frac{d}{dx}[f_1(x)f_2(x)f_3(x)f_4(x)]$. Also find a formula for $\frac{d}{dx}[f(x)]^4$.

80. (a) $\left(1, \frac{e}{2}\right)$ **(b)** $y = \frac{e}{2}$
(c) $-° < x < 1$ or $1 < x < °$
(d) Nowhere.
(e)



(f) f increasing on $(-°, 1)$ and $(1, \infty)$, where $f'(x) > 0$, and never decreasing (there is nowhere that $f'(x) < 0$).

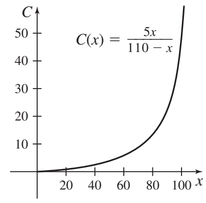
81. (a) $\frac{8}{5}$ **(b)** $-\frac{29}{10}$ **(c)** $\frac{1}{9}$
(d) $-\frac{19}{160}$ **(e)** $\frac{1}{9}$ **(f)** $\frac{12}{25}$

82. (a) $-\frac{8}{5}$ **(b)** -4 **(c)** 1
(d) $\frac{8}{15}$ **(e)** $\frac{6}{5}$ **(f)** $\frac{1}{2}$

83. (a) $v(t) = -9.8t + 39.2$ m/s
(b) $t = 4$ s **(c)** 78.4 m
(d) -9.8 m/s^2 **(e)** 8 s
(f) -39.2 m/s **(g)** 156.8 m

84. (a) 16 ft/s **(b)** $t = 2.5$ s
(c) 106.0 ft **(d)** ≈ -32 ft/s^2
(e) ≈ 5.074 s **(f)** ≈ -82.37 ft/s
(g) 206.0 ft

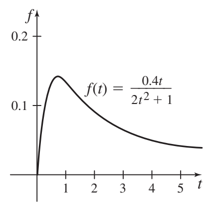
85. (a) $0 \le x \le 100$
(b)



(c) \$13,333.33
(d) $C'(x) = \frac{550}{(110-x)^2}$
(e) $C'(40) = 0.112 = \$112/\%$
$C'(60) = 0.220 = \$220/\%$
$C'(80) = 0.661 = \$661/\%$
$C'(90) = 1.375 = \$1,375/\%$
(f) Answers will vary. Sample answer: Cost to remove pollutant increases as concentration of pollutant increases. Cost to remove additional 1% of pollutant becomes much greater as concentration approaches 100%.

86. (a) $V'(t) = \frac{100t^2 - 50}{t^2}$ \$/yr
(b) $V'(1) = \$50$/yr
(c) $V'(3)$ ° \$94.44/yr
(d) Value of painting is appreciating faster after 3 yr than after 1 yr.

87. (a) $f'(t) = \frac{0.4 - 0.8t^2}{(2t^2+1)^2}$ mg/L/hr
(b) $f'\left(\frac{1}{6}\right) \approx 0.339$ mg/L/hr
$f'\left(\frac{1}{2}\right) \approx 0.089$ mg/L/hr
$f'(1)$ ° -0.044 mg/L/hr
(c) Answers will vary. Sample answer: Rate at which concentration of drug is increasing is less at 30 min than at 10 min, and at 1 h concentration is decreasing.
(d)



(e) Concentration highest at approx. 45 min and is about 0.14 mg/L.

88. (a) $P'(t) = \frac{1000(400 - 4t^2)}{(100+t^2)^2}$ bacteria/hr
(b) $P'(1)$ ° 38.8 bacteria/hr
$P'(2)$ ° 35.5 bacteria/hr
$P'(3)$ ° 30.6 bacteria/hr
$P'(4)$ ° 25.0 bacteria/hr

Answers continue on p. 206

107. If f and g are differentiable functions, show that

if $F(x)=\dfrac{1}{f(x)g(x)}$, then

$$F'(x)=-F(x)\left[\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right]$$

provided $f(x)\neq 0$, $g(x)\neq 0$.

108. Higher-Order Derivatives If $f(x)=\dfrac{1}{1-x}$, find a formula for the nth derivative of f. That is, find $f^{(n)}(x)$.

109. Let $f(x)=\dfrac{x^6-x^4+x^2}{x^4+1}$. Rewrite f in the form $(x^4+1)f(x)=x^6-x^4+x^2$. Now find $f'(x)$ without using the quotient rule.

110. If f and g are differentiable functions with $f\neq -g$, find the derivative of $\dfrac{fg}{f+g}$.

CAS **111.** $f(x)=\dfrac{2x}{x+1}$.

(a) Use technology to find $f'(x)$.

(b) Simplify f' to a single fraction using either algebra or a CAS.

(c) Use technology to find $f^{(5)}(x)$.
Hint: Your CAS may have a method for finding higher-order derivatives without finding other derivatives first.

Challenge Problems

112. Suppose f and g have derivatives up to the fourth order. Find the first four derivatives of the product fg and simplify the answers. In particular, show that the fourth derivative is

$$\frac{d^4}{dx^4}(fg)=f^{(4)}g+4f^{(3)}g^{(1)}+6f^{(2)}g^{(2)}+4f^{(1)}g^{(3)}+fg^{(4)}$$

Identify a pattern for the higher-order derivatives of fg.

113. Suppose $f_1(x),\ldots,f_n(x)$ are differentiable functions.

(a) Find $\dfrac{d}{dx}[f_1(x)\cdot\ldots\cdot f_n(x)]$.

(b) Find $\dfrac{d}{dx}\dfrac{1}{f_1(x)\cdot\ldots\cdot f_n(x)}$.

114. Let a, b, c, and d be real numbers. Define

$$\begin{vmatrix} a & b\\ c & d\end{vmatrix}=ad-bc$$

This is called a 2×2 **determinant** and it arises in the study of linear equations. Let $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ be differentiable and let

$$D(x)=\begin{vmatrix} f_1(x) & f_2(x)\\ f_3(x) & f_4(x)\end{vmatrix}$$

Show that

$$D'(x)=\begin{vmatrix} f_1'(x) & f_2'(x)\\ f_3(x) & f_4(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x)\\ f_3'(x) & f_4'(x)\end{vmatrix}$$

115. Let $f_0(x)=x-1$

$$f_1(x)=1+\frac{1}{x-1}$$

$$f_2(x)=1+\cfrac{1}{1+\cfrac{1}{x-1}}$$

$$f_3(x)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x-1}}}$$

(a) Write f_1, f_2, f_3, f_4, and f_5 in the form $\dfrac{ax+b}{cx+d}$.

(b) Using the results from (a), write the sequence of numbers representing the coefficients of x in the numerator, beginning with $f_0(x)=x-1$.

(c) Write the sequence in (b) as a recursive sequence.
Hint: Look at the sum of consecutive terms.

(d) Find f_0', f_1', f_2', f_3', f_4', and f_5'.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 197 **1.** What is the instantaneous rate of change at $x=-2$ of the function $f(x)=\dfrac{x-1}{x^2+2}$?

(A) $-\dfrac{1}{6}$ (B) $\dfrac{1}{9}$ (C) $\dfrac{1}{2}$ (D) -1

PAGE 197 **2.** An equation of the tangent line to the graph of $f(x)=\dfrac{5x-3}{3x-6}$ at the point $(3, 4)$ is

(A) $7x+3y=37$ (B) $7x+3y=33$
(C) $7x-3y=9$ (D) $13x+3y=51$

PAGE 197 **3.** If f, g, and h are nonzero differentiable functions of x, then $\dfrac{d}{dx}\left(\dfrac{gh}{f}\right)=$

(A) $\dfrac{fgh'+fg'h-f'gh}{f^2}$ (B) $\dfrac{g'h'-ghf'}{f^2}$
(C) $\dfrac{gh'+g'h}{f'}$ (D) $\dfrac{fgh'+fg'h+f'gh}{f^2}$

PAGE 195 **4.** If $y=x^3e^x$, then $\dfrac{dy}{dx}=$

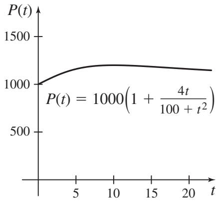
(A) $3x^2e^x$ (B) $3x^2+e^x$
(C) $3x^2e^x(x+1)$ (D) $x^2e^x(x+3)$

PAGE 198 **5.** $\dfrac{d}{dt}\left(t^2-\dfrac{1}{t^2}+\dfrac{1}{t}\right)$ at $t=2$ is

(A) $\dfrac{7}{2}$ (B) $\dfrac{9}{2}$ (C) $\dfrac{9}{4}$ (D) 4

(c) Rate of increase of population of bacteria decreases as time increases.

(d)



(e) Population apparently greatest at $t=10$ hr. $P(10)=1200$ bacteria.

89. (a) $-100{,}000\dfrac{2p+10}{(p^2+10p+50)^2}$ books/\$

(b) $D'(5)=-128$ books/\$
$D'(10)=-48$ books/\$
$D'(15)=-22.145$ books/\$

(c) Answers will vary. Sample answer: At \$5/book, increase of \$1 in price results in sales reduced by about 128 books. At \$10/book, increase of \$1 in price results in sales reduced by about 48 books. At \$15/book, increase of \$1 in price results in sales reduced by about 22.15 books. As price increases, demand becomes less sensitive to \$1 increase in price per book.

90. -2 lumens/m

91. (a) $\dfrac{dp}{dr}=-\dfrac{9nRT}{4\pi r^4}$ Pa/m

(b) As radius increases, pressure within container decreases.

(c) $p'\left({}^\circ\dfrac{1}{4}\right)\approx -415{,}945.358$ Pa/m

92. (a) $\dfrac{d\rho}{dr}=-\dfrac{9m}{4{}^\circ r^4}$

(b) As r increases, ρ decreases. As diver descends, r decreases, so ρ increases, and density of diver increases.

(c) ≈ -1397 kg/m^4

93. (a) $v(t)=3t^2-1$ m/s, $a(t)=6t$ m/s^2, $J(t)=6$ m/s^3, $S(t)=0$ m/s^4

(b) $t=\pm\dfrac{\sqrt{3}}{3}$ s

(c) $a(2)=12$ m/s^2, $a(5)=30$ m/s^2

(d) No.

(e) Answers will vary. Sample answer: Since snap S is always 0, object experiences constant jerk J.

94. (a) $v(t)=\dfrac{2}{3}t^3-2t+\dfrac{1}{2}$ m/s
$a(t)=2t^2-2$ m/s^2, $J(t)=4t$ m/s^3, $S(t)=4$ m/s^4

(b) $v(0)=1/2$ m/s, $v(3)=25/2$ m/s

(c) $a(0)=-2$ m/s^2. Initially, acceleration is negative, so velocity is initially decreasing.

(d) $J(t)=4t$, not constant. Interpretations will vary. Sample interpretation: Initially, $J=J(0)=0$, so acceleration is not increasing. As time increases, J is positive, so acceleration is increasing.

(e) Interpretations will vary. Sample interpretation: $S=4>0$, so jerk J is steadily increasing, so rate of increase of acceleration is increasing.

95. (a) $a(t)=1.6+1.998t$ m/s^2

(b) $J(t)=1.998$ m/s^3

96. (a) $a(t)=1.1+0.702t$ m/s^2

(b) $J(t)=0.702$ m/s^3

97. (a) $\dfrac{dJ}{dr}=-\dfrac{2I}{\pi r^3}$

(b) As radius increases, current density decreases.

(c) -1.273×10^{10} amps/m^3

98. See TSM for proof.

99. See TSM for proof.

100. $4x^3+12x^2-8x+4$

101. $6x^5-5x^4+20x^3-18x^2+2x-5$

Answers continue on p. 207

PAGE 201 **6.** The position of an object moving along a straight line at time t, in seconds, is given by $s(t) = 16t^2 - 5t + 20$ meters. What is the acceleration of the object when $t = 2$?

(A) 32 m/s (B) 0 m/s^2 (C) 32 m/s^2 (D) 64 m/s^2

PAGE 197 **7.** If $y = \frac{x-3}{x+3}$, $x \neq -3$, the instantaneous rate of change of y with respect to x at $x = 3$ is

(A) $-\frac{1}{6}$ (B) $\frac{1}{6}$ (C) $\frac{1}{36}$ (D) 1

PAGE 197 **8.** Find an equation of the normal line to the graph of the function $f(x) = \frac{x^2}{x+1}$ at $x = 1$.

(A) $8x + 6y = 11$ (B) $-8x + 6y = -5$
(C) $-3x + 4y = -1$ (D) $3x + 4y = 5$

PAGE 200 **9.** If $y = xe^x$, then the nth derivative of y is

(A) e^x (B) $(x+n)e^x$ (C) ne^x (D) $x^n e^x$

2.5 The Derivative of the Trigonometric Functions

OBJECTIVE ***When you finish this section, you should be able to:***

1 Differentiate trigonometric functions (p. 207)

1 Differentiate Trigonometric Functions

To find the derivatives of $y = \sin x$ and $y = \cos x$, we use the limits

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

that were established in Section 1.4.

THEOREM Derivative of $y = \sin x$

The derivative of $y = \sin x$ is $y' = \cos x$. That is,

$$y' = \frac{d}{dx}\sin x = \cos x$$

Proof

$$y' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{The definition of a derivative}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \qquad \sin(A+B) = \sin A \cos B + \sin B \cos A$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h}\right] \qquad \text{Rearrange terms.}$$

$$= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cdot \cos x\right] \qquad \text{Factor.}$$

$$= \left[\lim_{h \to 0} \sin x\right]\left[\lim_{h \to 0} \frac{\cos h - 1}{h}\right] + \left[\lim_{h \to 0} \cos x\right]\left[\lim_{h \to 0} \frac{\sin h}{h}\right] \qquad \text{Use properties of limits.}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x \qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0; \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad ■$$

NEED TO REVIEW? The trigonometric functions are discussed in Section P.6, pp. 52–58. Trigonometric identities are discussed in Appendix A.4, pp. A33 to A36.

The geometry of the derivative $\frac{d}{dx}\sin x = \cos x$ is shown in Figure 30. On the graph of $f(x) = \sin x$, the horizontal tangents are marked as well as the tangent lines that have slopes of 1 and -1. The derivative function is plotted on the second graph, and those points are connected with a smooth curve.

102. $12x^3(x^4+1)^2$

103. $9x^2(x^3+1)^2$

104. $3 - \frac{1}{x^2} - \frac{12}{x^5} - \frac{20}{x^6} - \frac{6}{x^7}$

105. $\frac{1}{x^2}\left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\left(\frac{3}{x^2}\right) + \left(1-\frac{1}{x}\right)\left(\frac{2}{x}\right)\left(1-\frac{1}{x^3}\right) + \left(1-\frac{1}{x^2}\right)\left(1-\frac{1}{x^3}\right)\right]$
$= \frac{1}{x^2} + \frac{2}{x^3} - \frac{4}{x^5} - \frac{5}{x^6} + \frac{6}{x^7}$

106. $\frac{d}{dx}[f_1(x)f_2(x)f_3(x)f_4(x)]$
$= f_1(x)f_2(x)f_3(x)f_4'(x)$
$+ f_1(x)f_2(x)f_3'(x)f_4(x)$
$+ f_1(x)f_2'(x)f_3(x)f_4(x)$
$+ f_1'(x)f_2(x)f_3(x)f_4(x)$;

$\frac{d}{dx}[f_1(x)]^4 = 4[f_1(x)]^3 f_1'(x)$

107. See TSM for proof.

108. $\frac{n!}{(1-x)^{n+1}}$

Answers continue on p. AA-4

TRM Alternate Examples Section 2.5

You can find the Alternate Examples for this section in PDF format in the Teacher's Resource Materials.

TRM AP® Calc Skill Builders Section 2.5

You can find the AP® Calc Skill Builders for this section in PDF format in the Teacher's Resource Materials.

COMMON ERRORS & MISCONCEPTIONS

Whenever students are using their calculators to approximate a derivative of a function that involves a trigonometric expression, remind them that they must be in radian mode or they will get the wrong answer. This is because $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ in radians, but if θ is measured in degrees, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{\pi}{180} = 0.01745$. If students have their calculator in degree mode, they will get an incorrect answer for any values (including derivatives) that involve trigonometry.

AP® EXAM TIP

Students have to know the sum identities for sin x and cos x to prove the derivative rules for these functions, but sum and difference identities do not usually show up on the AP® Exam. For a review of trigonometric identities, see Appendix A.4. There is not a specific list of identities that students must know for the AP® Exam, but Calculus AB students should know at least their Quotient Identities, Pythagorean Identities, and Double-Angle Identities. In addition to these, Calculus BC students should know their Power-Reducing Identities. While other identities might appear on the exam, these are the most common.

TRM Section 2.5: Worksheet 1

This worksheet walks the student through a graphical explanation of the derivative of the functions $y = \sin x$ and $y = \cos x$. The students are also asked to recall the 6 trigonometric derivatives from memory and to find the derivative of 2 trigonometric functions.

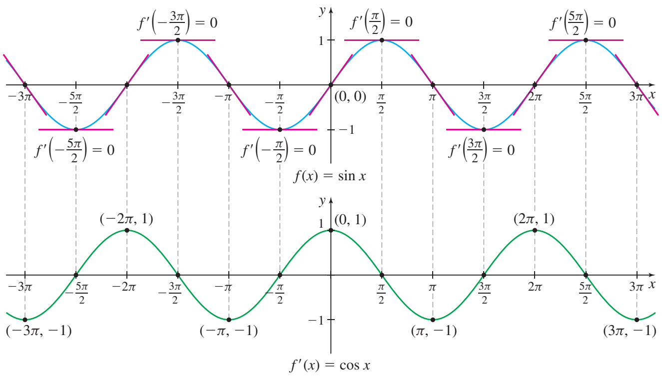


Figure 30

To find derivatives involving the trigonometric functions, use the sum, difference, product, and quotient rules and the derivative formulas from Sections 2.3 and 2.4.

EXAMPLE 1 Differentiating the Sine Function

Find y' if:

(a) $y=x+4\sin x$ **(b)** $y=x^2\sin x$ **(c)** $y=\dfrac{\sin x}{x}$ **(d)** $y=e^x\sin x$

Solution

(a) Use the Sum Rule and the Constant Multiple Rule.

$$y'=\frac{d}{dx}(x+4\sin x)=\frac{d}{dx}x+\frac{d}{dx}(4\sin x)=1+4\frac{d}{dx}\sin x=1+4\cos x$$

(b) Use the Product Rule.

$$y'=\frac{d}{dx}(x^2\sin x)=x^2\left[\frac{d}{dx}\sin x\right]+\left[\frac{d}{dx}x^2\right]\sin x=x^2\cos x+2x\sin x$$

(c) Use the Quotient Rule.

$$y'=\frac{d}{dx}\left(\frac{\sin x}{x}\right)=\frac{\left[\frac{d}{dx}\sin x\right]\cdot x-\sin x\cdot\left[\frac{d}{dx}x\right]}{x^2}=\frac{x\cos x-\sin x}{x^2}$$

(d) Use the Product Rule.

$$y'=\frac{d}{dx}(e^x\sin x)=e^x\frac{d}{dx}\sin x+\left(\frac{d}{dx}e^x\right)\sin x$$
$$=e^x\cos x+e^x\sin x=e^x(\cos x+\sin x)$$ ■

NOW WORK Problems 5, 29 and AP® Practice Problems 1, 5 and 10.

THEOREM Derivative of $y=\cos x$

The derivative of $y=\cos x$ is

$$y'=\frac{d}{dx}\cos x=-\sin x$$

You are asked to prove this in Problem 75.

AP® CALC SKILL BUILDER FOR EXAMPLE 1

Differentiating the Sine Function

Find y' if

(a) $y=\dfrac{1}{x^2+1}+3\sin x$

(b) $y=2x\sin x$

Solution

(a) $y'=-\dfrac{\frac{d}{dx}(x^2+1)}{(x^2+1)^2}+3\cos x$

$=-\dfrac{2x}{(x^2+1)^2}+3\cos x$

(b) $y'=(2)(\sin x)+(2x)(\cos x)$

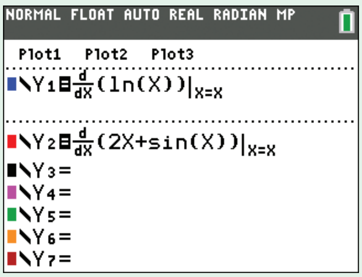
$y'=2\sin x+2x\cos x$

GRAPHING CALCULATOR PRACTICE

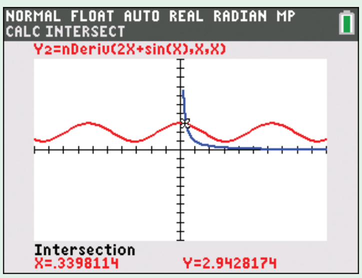
Find all values of x where the lines tangent to $y=\ln x$ and $y=2x+\sin x$ have the same slope.

Solution

Since students do not currently know how to take the derivative of $y=\ln x$, they will have to use a calculator to help them answer this question. First, graph the derivative of each function using your calculator's numerical derivative feature.



Next, use the intersection feature of your calculator to find the point of intersection of the derivatives.



Since the derivatives intersect at $x=0.3398114$, the two functions will both have the same slope at $x=0.3398114$. Thus, the slope of the tangent lines will be equal at $x=0.340$.

Building Calculator Skills

You can often have students practice multiple calculator skills at the same time. See the Graphing Calculator Practice at left for an example of how you can have students practice their ability to find a numerical derivative and the intersection of two curves in the same problem.

AP® EXAM TIP

Students should be able to recognize the definition of a derivative and use it to evaluate a limit. For example, if students are asked to evaluate $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin(x)}{h}$, they should recognize that the limit is in the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, which equals $f'(x)$. So if $f(x)=\sin x$, $f'(x)=\cos x$. Thus, $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin(x)}{h}=\cos x$.

CALC CLIP

EXAMPLE 2 Differentiating Trigonometric Functions

Find the derivative of each function:

(a) $f(x) = x^2 \cos x$ **(b)** $g(\theta) = \dfrac{\cos\theta}{1 - \sin\theta}$ **(c)** $F(t) = \dfrac{e^t}{\cos t}$

Solution

(a)
$$f'(x) = \frac{d}{dx}(x^2 \cos x) = x^2 \frac{d}{dx}\cos x + \left(\frac{d}{dx}x^2\right)(\cos x)$$
$$= x^2(-\sin x) + 2x \cos x = 2x \cos x - x^2 \sin x$$

(b)
$$g'(\theta) = \frac{d}{d\theta}\left(\frac{\cos\theta}{1-\sin\theta}\right) = \frac{\left(\dfrac{d}{d\theta}\cos\theta\right)(1-\sin\theta) - (\cos\theta)\left[\dfrac{d}{d\theta}(1-\sin\theta)\right]}{(1-\sin\theta)^2}$$
$$= \frac{-\sin\theta\,(1-\sin\theta) - \cos\theta(-\cos\theta)}{(1-\sin\theta)^2} = \frac{-\sin\theta + \sin^2\theta + \cos^2\theta}{(1-\sin\theta)^2}$$
$$= \frac{-\sin\theta + 1}{(1-\sin\theta)^2} = \frac{1}{1-\sin\theta}$$

(c)
$$F'(t) = \frac{d}{dt}\left(\frac{e^t}{\cos t}\right) = \frac{\left(\dfrac{d}{dt}e^t\right)(\cos t) - e^t\left(\dfrac{d}{dt}\cos t\right)}{\cos^2 t} = \frac{e^t \cos t - e^t(-\sin t)}{\cos^2 t}$$
$$= \frac{e^t(\cos t + \sin t)}{\cos^2 t}$$ ■

NOW WORK Problem 13 and AP Practice® Problems 2, 6 and 8.

EXAMPLE 3 Identifying Horizontal Tangent Lines

Find all points on the graph of $f(x) = x + \sin x$ where the tangent line is horizontal.

Solution

Since tangent lines are horizontal at points on the graph of f where $f'(x) = 0$, begin by finding $f'(x) = 1 + \cos x$. Now solve the equation:

$$f'(x) = 1 + \cos x = 0$$
$$\cos x = -1$$
$$x = (2k+1)\pi$$

where k is an integer.

Since $\sin[(2k+1)\pi] = 0$, then $f((2k+1)\pi) = (2k+1)\pi$. So, at each of the points $((2k+1)\pi, (2k+1)\pi)$, the graph of f has a horizontal tangent line. See Figure 31. ■

Notice in Figure 31 that each of the points with a horizontal tangent line lies on the line $y = x$.

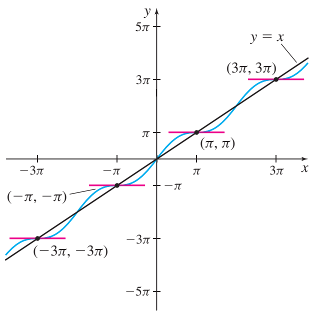


Figure 31 $f(x) = x + \sin x$

NOW WORK Problem 57 and AP® Practice Problem 9.

The derivatives of the remaining four trigonometric functions are obtained using trigonometric identities and basic derivative rules. We establish the formula for the derivative of $y = \tan x$ in Example 4. You are asked to prove formulas for the derivative of the secant function, the cosecant function, and the cotangent function in the exercises. (See Problems 76–78.)

ALTERNATE EXAMPLE

Differentiating Trigonometric Functions

Find the first eight derivatives of $y = \cos x$. Then find the fiftieth derivative of $y = \cos x$.

Solution

$y = \cos x$
$y' = -\sin x$
$y'' = -\cos x$
$y''' = \sin x$
$y^{(4)} = \cos x$
$y^{(5)} = -\sin x$
$y^{(6)} = -\cos x$
$y^{(7)} = \sin x$
$y^{(8)} = \cos x$
...
$y^{(48)} = \cos x$; therefore, $y^{(50)} = -\cos x$

Teaching Tip

For the Alternate Example above, see if students can identify the pattern in the derivatives of cosine and use it to find the fiftieth derivative of $\cos x$.

AP® CALC SKILL BUILDER FOR EXAMPLE 3

Identifying Horizontal Tangent Lines

Find all points on the graph of $f(x) = 2x + 4\cos x$ where the tangent line is horizontal.

Solution

To find the tangent line, first find the derivative of f:

$$f'(x) = 2 - 4\sin x$$

Now solve $f'(x) = 0$.

$$2 - 4\sin x = 0$$
$$\sin x = \frac{1}{2}$$

Since $\sin x = \dfrac{1}{2}$ for $x = \dfrac{\pi}{6} \pm 2k\pi$ and $x = \dfrac{5\pi}{6} \pm 2k\pi$, then the graph of f has horizontal tangents at $x = \dfrac{\pi}{6} \pm 2k\pi$ and $x = \dfrac{5\pi}{6} \pm 2k\pi$.

AP® CALC SKILL BUILDER FOR EXAMPLE 2

Differentiating Trigonometric Functions

Find y' if

(a) $y = \dfrac{1}{x} + 2\cos x$

(b) $y = \dfrac{\sin x}{\cos x}$

Solution

(a) $y' = -\dfrac{1}{x^2} - 2\sin x$

(b) $y' = \dfrac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$y' = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$y' = \dfrac{1}{\cos^2 x}$

$y' = \sec^2 x$

AP® CALC SKILL BUILDER FOR EXAMPLE 4

Differentiating $y = \tan x$

Find $\dfrac{d}{dx}\left(\dfrac{\tan x}{x}\right)$.

Solution

$$\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{(\sec^2 x)(x) - (\tan x)(1)}{x^2} = \frac{x\sec^2 x - \tan x}{x^2}$$

AP® CALC SKILL BUILDER FOR EXAMPLE 5

Finding Acceleration

The position x (in meters) of an object moving along a line at time t (in seconds), $0 \le t \le \dfrac{\pi}{2}$, is given by $x(t) = 8\cos(t) + 2t^2 - 1$. What is the velocity of the object when its acceleration is 0?

Solution

First, we find expressions for the velocity and acceleration of the object by taking the first and second derivatives of x.

$$v(t) = x'(t) = -8\sin(t) + 4t$$
$$a(t) = v'(t) = -8\cos(t) + 4$$

Now we solve $a(t) = 0$:

$$-8\cos(t) + 4 = 0$$
$$\cos(t) = \frac{1}{2}$$
$$t = \frac{\pi}{3}$$

The velocity at $t = \dfrac{\pi}{3}$ is:

$$v\left(\frac{\pi}{3}\right) = -8\sin\left(\frac{\pi}{3}\right) + 4\left(\frac{\pi}{3}\right) = -8\left(\frac{\sqrt{3}}{2}\right) + \frac{4\pi}{3} = -4\sqrt{3} + \frac{4\pi}{3} \approx -2.739$$

The velocity of the object when the acceleration is 0 is approximately −2.739 m/s.

TRM Section 2.5: Worksheet 2

This worksheet contains 3 trigonometric functions. The students are asked to find the second derivative of each function.

EXAMPLE 4 Differentiating $y = \tan x$

Show that the derivative of $y = \tan x$ is

$$y' = \frac{d}{dx}\tan x = \sec^2 x$$

Solution

$$y' = \frac{d}{dx}\tan x \underset{\uparrow\ \text{Identity}}{=} \frac{d}{dx}\frac{\sin x}{\cos x} \underset{\uparrow\ \text{Quotient Rule}}{=} \frac{\left[\frac{d}{dx}\sin x\right]\cos x - \sin x\left[\frac{d}{dx}\cos x\right]}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \quad ■$$

NOW WORK Problem 15 and AP® Practice Problems 3 and 7.

Table 4 lists the derivatives of the six trigonometric functions along with the domain of each derivative.

TABLE 4

Derivative Function	Domain of the Derivative Function
$\frac{d}{dx}\sin x = \cos x$	$(-\infty, \infty)$
$\frac{d}{dx}\cos x = -\sin x$	$(-\infty, \infty)$
$\frac{d}{dx}\tan x = \sec^2 x$	$\left\{x \mid x \neq \frac{2k+1}{2}\pi,\ k \text{ an integer}\right\}$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\{x \mid x \neq k\pi,\ k \text{ an integer}\}$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\{x \mid x \neq k\pi,\ k \text{ an integer}\}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\left\{x \mid x \neq \frac{2k+1}{2}\pi,\ k \text{ an integer}\right\}$

NOTE If the trigonometric function begins with the letter *c*, that is, cosine, cotangent, or cosecant, then its derivative has a minus sign.

NOW WORK Problem 35.

EXAMPLE 5 Finding the Second Derivative of a Trigonometric Function

Find $f''\left(\frac{\pi}{4}\right)$ if $f(x) = \sec x$.

Solution

If $f(x) = \sec x$, then $f'(x) = \sec x \tan x$ and

$$f''(x) = \frac{d}{dx}(\sec x \tan x) \underset{\uparrow\ \text{Use the Product Rule.}}{=} \sec x\left(\frac{d}{dx}\tan x\right) + \left(\frac{d}{dx}\sec x\right)\tan x$$

$$= \sec x \cdot \sec^2 x + (\sec x \tan x)\tan x = \sec^3 x + \sec x \tan^2 x$$

$$f''\left(\frac{\pi}{4}\right) = \sec^3\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) \underset{\uparrow\ \sec\frac{\pi}{4} = \sqrt{2};\ \tan\frac{\pi}{4} = 1}{=} \left(\sqrt{2}\right)^3 + \sqrt{2} \cdot 1^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2} \quad ■$$

NOW WORK Problem 45 and AP® Practice Problem 4.

Teaching Tip

Spend a few minutes with your students reciting the derivatives of the 6 trigonometric functions. Read them out loud two times slowly. Point out to the students that each trigonometric function that starts with the letter *c* has a negative sign in the derivative formula. Point out the patterns in the similarities of the derivatives (i.e., the derivative of sec *x* and csc *x* are similar). Ask the students to recite aloud the derivatives in unison. Do so in order twice, then begin to switch the order around. A few minutes spent working on memorizing these six derivatives will go a long way.

Teaching Tip

If you would like to show your students how to derive the formulas for the derivatives of sec *x*, csc *x*, and cot *x*, each of the formulas can be found by writing the trigonometric function as a fraction and then using the Quotient Rule to find the derivative, similar to Example 4.

Application: Simple Harmonic Motion

Simple harmonic motion is a repetitive motion that can be modeled by a trigonometric function. A swinging pendulum and an oscillating spring are examples of simple harmonic motion.

EXAMPLE 6 Analyzing Simple Harmonic Motion

An object hangs on a spring, making the spring 2 m long in its equilibrium position. See Figure 32. If the object is pulled down 1 m and released, it oscillates up and down. The length l of the spring after t seconds is modeled by the function $l(t) = 2 + \cos t$.

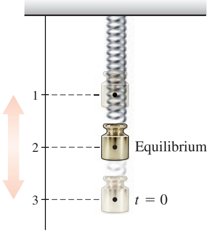


Figure 32

(a) How does the length of the spring vary?
(b) Find the velocity of the object.
(c) At what position is the speed of the object a maximum?
(d) Find the acceleration of the object.
(e) At what position is the acceleration equal to 0?

Solution

(a) Since $l(t) = 2 + \cos t$ and $-1 \le \cos t \le 1$, the length of the spring varies between 1 and 3 m.

(b) The velocity v of the object is

$$v = l'(t) = \frac{d}{dt}(2 + \cos t) = -\sin t$$

(c) Speed is the magnitude of velocity. Since $v = -\sin t$, the speed of the object is $|v| = |-\sin t| = |\sin t|$. Since $-1 \le \sin t \le 1$, the object moves the fastest when $|v| = |\sin t| = 1$. This occurs when $\sin t = \pm 1$ or, equivalently, when $\cos t = 0$. So, the speed is a maximum when $l(t) = 2$, that is, when the spring is at the equilibrium position.

(d) The acceleration a of the object is given by

$$a = l''(t) = \frac{d}{dt}l'(t) = \frac{d}{dt}(-\sin t) = -\cos t$$

(e) Since $a = -\cos t$, the acceleration is zero when $\cos t = 0$. So, $a = 0$ when $l(t) = 2$, that is, when the spring is at the equilibrium position. This is the same time at which the speed is maximum. ■

Figure 33 shows the graphs of the length of the spring $y = l(t)$, the velocity $y = v(t)$, and the acceleration $y = a(t)$.

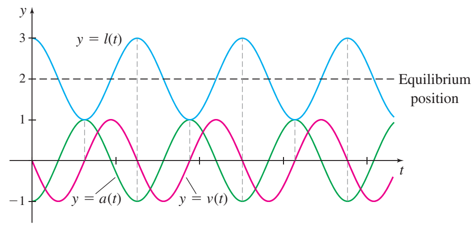


Figure 33 $y = l(t)$(blue), $y = v(t)$(red), $y = a(t)$(green)

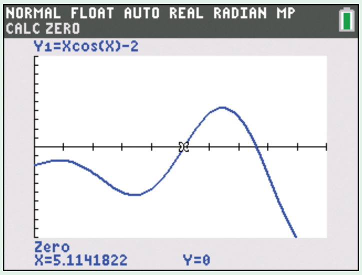
NOW WORK Problem 65.

GRAPHING CALCULATOR PRACTICE

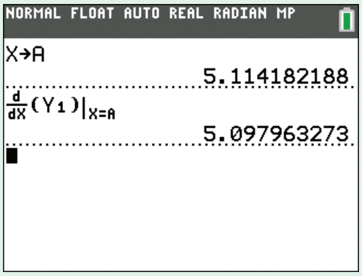
The velocity of a particle is given by $v(t) = t\cos t - 2$, where $t \ge 0$. Find the acceleration of the particle the first time that the velocity is equal to 0.

Solution

First, graph the velocity function and find the first time it is equal to 0. It can be helpful to have students set the Xmin value in their window to 0 to match the domain so they do not accidentally choose a solution not in the domain.



The calculator stores this value as X, so you can store it and use it to find the acceleration at $x = 5.114$, which will be the derivative of the velocity at $x = 5.114$.



The first time that the particle's velocity is 0, its acceleration will be 5.098.

TRM Section 2.5: Worksheet 3

This worksheet contains 2 trigonometric functions and their corresponding graphs. The students are asked to find the tangent line at the given point and sketch the tangent line on the graph.

Building Calculator Skills

Always have students store intermediate values in their calculator instead of writing down a rounded answer. Using a rounded answer in an intermediate step can cause students to get an incorrect final answer.

Teaching Tip

Consider using College Board's Personal Progress Check for Unit 2 either for homework or on a review day. This progress check includes approximately 30 multiple-choice questions and 3 partial or full free-response questions.

MUST-DO PROBLEMS FOR EXAM READINESS

AB: 5, 7, 11, 15, 19, 21, 29, 37, 41, 45, 55, 57, and all AP® Practice Problems

BC: 5, 11, 13, 15, 23, 29, 31, 35, 45, 57, 63, 65, and all AP® Practice Problems

TRM Full Solutions to Section 2.5 Problems and 2.5 AP® Practice Problems

2.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\dfrac{d}{dx}\cos x = \sin x$
2. *True or False* $\dfrac{d}{dx}\tan x = \cot x$
3. *True or False* $\dfrac{d^2}{dx^2}\sin x = -\sin x$
4. *True or False* $\dfrac{d}{dx}\sin\dfrac{\pi}{3} = \cos\dfrac{\pi}{3}$

Skill Building

In Problems 5–38, find y'.

PAGE 208 **5.** $y = x - \sin x$
6. $y = \cos x - x^2$
7. $y = \tan x + \cos x$
8. $y = \sin x - \tan x$
9. $y = 3\sin\theta - 2\cos\theta$
10. $y = 4\tan\theta + \sin\theta$
11. $y = \sin x \cos x$
12. $y = \cot x \tan x$
PAGE 209 **13.** $y = t\cos t$
14. $y = t^2 \tan t$
PAGE 210 **15.** $y = e^x \tan x$
16. $y = e^x \sec x$
17. $y = \pi \sec u \tan u$
18. $y = \pi u \tan u$
19. $y = \dfrac{\cot x}{x}$
20. $y = \dfrac{\csc x}{x}$
21. $y = x^2 \sin x$
22. $y = t^2 \tan t$
23. $y = t\tan t - \sqrt{3}\sec t$
24. $y = x\sec x + \sqrt{2}\cot x$
25. $y = \dfrac{\sin\theta}{1-\cos\theta}$
26. $y = \dfrac{x}{\cos x}$
27. $y = \dfrac{\sin t}{1+t}$
28. $y = \dfrac{\tan u}{1+u}$
PAGE 208 **29.** $y = \dfrac{\sin x}{e^x}$
30. $y = \dfrac{\cos x}{e^x}$
31. $y = \dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}$
32. $y = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$
33. $y = \dfrac{\sec t}{1 + t\sin t}$
34. $y = \dfrac{\csc t}{1 + t\cos t}$
PAGE 210 **35.** $y = \csc\theta\cot\theta$
36. $y = \tan\theta\cos\theta$
37. $y = \dfrac{1+\tan x}{1-\tan x}$
38. $y = \dfrac{\csc x - \cot x}{\csc x + \cot x}$

In Problems 39–50, find y''.

39. $y = \sin x$
40. $y = \cos x$
41. $y = \tan\theta$
42. $y = \cot\theta$
43. $y = t\sin t$
44. $y = t\cos t$
PAGE 210 **45.** $y = e^x \sin x$
46. $y = e^x \cos x$
47. $y = 2\sin u - 3\cos u$
48. $y = 3\sin u + 4\cos u$
49. $y = a\sin x + b\cos x$
50. $y = a\sec\theta + b\tan\theta$

In Problems 51–56:
(a) Find an equation of the tangent line to the graph of f at the indicated point.
(b) Graph the function and the tangent line.

51. $f(x) = \sin x$ at $(0, 0)$
52. $f(x) = \cos x$ at $\left(\dfrac{\pi}{3}, \dfrac{1}{2}\right)$
53. $f(x) = \tan x$ at $(0, 0)$
54. $f(x) = \tan x$ at $\left(\dfrac{\pi}{4}, 1\right)$
55. $f(x) = \sin x + \cos x$ at $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$
56. $f(x) = \sin x - \cos x$ at $\left(\dfrac{\pi}{4}, 0\right)$

In Problems 57–60:
(a) Find all points on the graph of f where the tangent line is horizontal.
(b) Graph the function and the horizontal tangent lines on the interval $[-2\pi, 2\pi]$.

PAGE 209 **57.** $f(x) = 2\sin x + \cos x$
58. $f(x) = \cos x - \sin x$
59. $f(x) = \sec x$
60. $f(x) = \csc x$

Applications and Extensions

In Problems 61 and 62, find the nth derivative of each function.

61. $f(x) = \sin x$
62. $f(\theta) = \cos\theta$

63. What is $\lim\limits_{h\to 0}\dfrac{\cos\left(\dfrac{\pi}{2}+h\right) - \cos\dfrac{\pi}{2}}{h}$?

64. What is $\lim\limits_{h\to 0}\dfrac{\sin(\pi+h) - \sin\pi}{h}$?

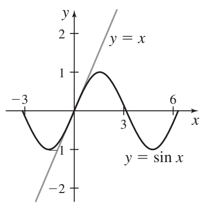
PAGE 211 **65. Simple Harmonic Motion** The signed distance s (in meters) of an object from the origin at time t (in seconds) is modeled by the position function $s(t) = \dfrac{1}{8}\cos t$.

(a) Find the velocity $v = v(t)$ of the object.
(b) When is the speed of the object a maximum?
(c) Find the acceleration $a = a(t)$ of the object.
(d) When is the acceleration equal to 0?
(e) Graph s, v, and a on the same screen.

Answers to Section 2.5 Problems

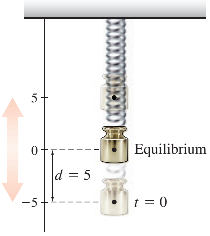
1. False. **2.** False. **3.** True. **4.** False. **5.** $1-\cos x$ **6.** $-(\sin x + 2x)$ **7.** $\sec^2 x - \sin x$ **8.** $\cos x - \sec^2 x$ **9.** $3\cos\theta + 2\sin\theta$ **10.** $4\sec^2\theta + \cos\theta$ **11.** $\cos^2 x - \sin^2 x$ or $\cos 2x$ **12.** 0 **13.** $\cos t - t\sin t$ **14.** $t^2\sec^2 t + 2t\tan t$ **15.** $e^x(\sec^2 x + \tan x)$ **16.** $e^x \sec x(\tan x + 1)$ **17.** $\pi\sec u(\sec^2 u + \tan^2 u)$ **18.** $\pi(u\sec^2 u + \tan u)$ **19.** $-\dfrac{x\csc^2 x + \cot x}{x^2}$ **20.** $-\dfrac{x\csc x\cot x + \csc x}{x^2}$ **21.** $x(x\cos x + 2\sin x)$ **22.** $t^2\sec^2 t + 2t\tan t$ **23.** $t\sec^2 t + \tan t - \sqrt{3}\sec t\tan t$ **24.** $x\sec x\tan x + \sec x - \sqrt{2}\csc^2 x$ **25.** $\dfrac{1}{\cos\theta - 1}$ **26.** $\dfrac{\cos x + x\sin x}{\cos^2 x}$ **27.** $\dfrac{\cos t + t\cos t - \sin t}{(1+t)^2}$ **28.** $\dfrac{(1+u)\sec^2 u - \tan u}{(1+u)^2}$ **29.** $\dfrac{\cos x - \sin x}{e^x}$ **30.** $\dfrac{-\sin x - \cos x}{e^x}$ **31.** $-\dfrac{2}{(\sin\theta - \cos\theta)^2}$ **32.** $\dfrac{2}{(\sin\theta + \cos\theta)^2}$ **33.** $\dfrac{\sec t\tan t + t\tan^2 t - t - \tan t}{(1+t\sin t)^2}$ **34.** $-\dfrac{\csc t\cot t - t\cot^2 t - \cot t + t}{(1+t\cos t)^2}$ **35.** $-\csc\theta(\csc^2\theta + \cot^2\theta)$ **36.** $\cos\theta$ **37.** $\dfrac{2\sec^2 x}{(1-\tan x)^2}$ **38.** $\dfrac{2\sin x}{(1+\cos x)^2}$ **39.** $-\sin x$ **40.** $-\cos x$ **41.** $2\tan\theta\sec^2\theta$ **42.** $\sec\theta(\sec^2\theta + \tan^2\theta)$ **43.** $2\cos t - t\sin t$ **44.** $-(2\sin t + t\cos t)$ **45.** $2e^x\cos x$ **46.** $-2e^x\sin x$ **47.** $3\cos u - 2\sin u$ **48.** $-(3\sin u + 4\cos u)$ **49.** $-(a\sin x + b\cos x)$ **50.** $a\sec\theta(\sec^2\theta + \tan^2\theta) + 2b\sec^2\theta\tan\theta$ **51.** (a) $y = x$

(b)



Answers continue on p. 213

66. Simple Harmonic Motion An object attached to a coiled spring is pulled down a distance $d = 5$ cm from its equilibrium position and then released as shown in the figure. The motion of the object at time t seconds is simple harmonic and is modeled by $d(t) = -5\cos t$.

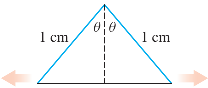


(a) As t varies from 0 to 2π, how does the length of the spring vary?
(b) Find the velocity $v = v(t)$ of the object.
(c) When is the speed of the object a maximum?
(d) Find the acceleration $a = a(t)$ of the object.
(e) When is the acceleration equal to 0?
(f) Graph d, v, and a on the same set of axes.

67. Rate of Change A large, 8-ft-high decorative mirror is placed on a wood floor and leaned against a wall. The weight of the mirror and the slickness of the floor cause the mirror to slip.

(a) If θ is the angle between the top of the mirror and the wall, and y is the distance from the floor to the top of the mirror, what is the rate of change of y with respect to θ?
(b) In feet/radian, how fast is the top of the mirror slipping down the wall when $\theta = \frac{\pi}{4}$?

68. Rate of Change The sides of an isosceles triangle are sliding outward. See the figure.



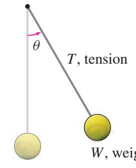
(a) Find the rate of change of the area of the triangle with respect to θ.
(b) How fast is the area changing when $\theta = \frac{\pi}{6}$?

69. Sea Waves Waves in deep water tend to have the symmetric form of the function $f(x) = \sin x$. As they approach shore, however, the sea floor creates drag, which changes the shape of the wave. The trough of the wave widens and the height of the wave increases, so the top of the wave is no longer symmetric with the trough. This type of wave can be represented by a function such as

$$w(x) = \frac{4}{2 + \cos x}$$

(a) Graph $w = w(x)$ for $0 \le x \le 4\pi$.
(b) What is the maximum and the minimum value of w?
(c) Find the values of x, $0 < x < 4\pi$, at which $w'(x) = 0$.
(d) Evaluate w' near the peak at π, using $x = \pi - 0.1$, and near the trough at 2π, using $x = 2\pi - 0.1$.
(e) Explain how these values confirm a nonsymmetric wave shape.

70. Swinging Pendulum A simple pendulum is a small-sized ball swinging from a light string. As it swings, the supporting string makes an angle θ with the vertical. See the figure. At an angle θ, the tension in the string is $T = \frac{W}{\cos\theta}$, where W is the weight of the swinging ball.



(a) Find the rate of change of the tension T with respect to θ when the pendulum is at its highest point ($\theta = \theta_{max}$).
(b) Find the rate of change of the tension T with respect to θ when the pendulum is at its lowest point.
(c) What is the tension at the lowest point?

71. Restaurant Sales A restaurant in Naples, Florida, is very busy during the winter months and extremely slow over the summer. But every year the restaurant grows its sales. Suppose over the next two years, the revenue R, in units of \$10,000, is projected to follow the model

$$R = R(t) = \sin t + 0.3t + 1 \qquad 0 \le t \le 12$$

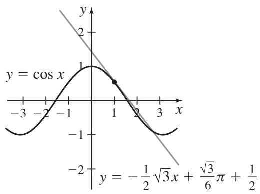
where $t = 0$ corresponds to November 1, 2018; $t = 1$ corresponds to January 1, 2019; $t = 2$ corresponds to March 1, 2019; and so on.

(a) What is the projected revenue for November 1, 2018; March 1, 2019; September 1, 2019; and January 1, 2020?
(b) What is the rate of change of revenue with respect to time?
(c) What is the rate of change of revenue with respect to time for January 1, 2020?
(d) Graph the revenue function and the derivative function $R' = R'(t)$.
(e) Does the graph of R support the facts that every year the restaurant grows its sales and that sales are higher during the winter and lower during the summer? Explain.

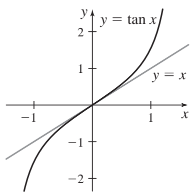
72. Polarizing Sunglasses Polarizing sunglasses are filters that transmit only light for which the electric field oscillations are in a specific direction. Light is polarized naturally by scattering off the molecules in the atmosphere and by reflecting off many (but not all) types of surfaces. If light of intensity I_0 is already polarized in a certain direction, and the transmission direction of the polarizing filter makes an angle with that direction, then the intensity I of the light after passing through the filter is given by **Malus's Law**, $I(\theta) = I_0 \cos^2\theta$.


REUTERS/Alamy

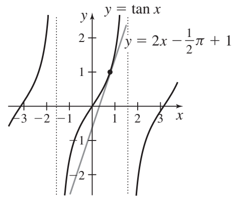
52. (a) $y = -\frac{1}{2}\sqrt{3}x + \frac{\sqrt{3}}{6}\pi + \frac{1}{2}$
(b)



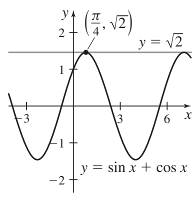
53. (a) $y = x$
(b)



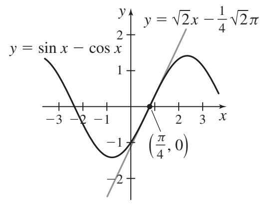
54. (a) $y = 2x - \frac{1}{2}\pi + 1$
(b)



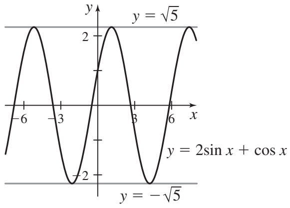
55. (a) $y = \sqrt{2}$
(b)



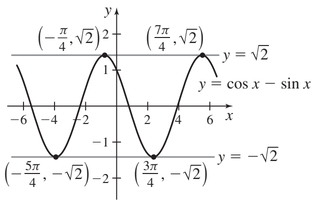
56. (a) $y = \sqrt{2}x - \frac{\pi}{4}\sqrt{2}$
(b)



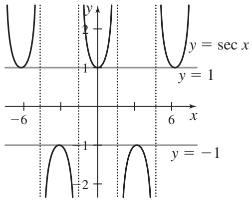
57. (a) $\{(\tan^{-1}2 + 2n\pi, \sqrt{5}) \mid n \text{ is an integer}\}$, $\{(\tan^{-1}2 + (2n+1)\pi, -\sqrt{5}) \mid n \text{ is an integer}\}$
(b)



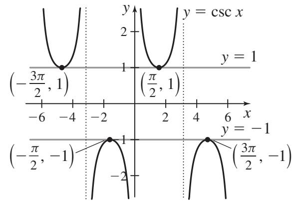
58. (a) $\left\{\left(-\frac{\pi}{4} + n\pi, (-1)^n\sqrt{2}\right) \middle| n \text{ is an integer}\right\}$
(b)



59. (a) $\{(2n\pi, 1) \mid n \text{ is an integer}\}$, $\{((2n+1)\pi, -1) \mid n \text{ is an integer}\}$
(b)



60. (a) $\left\{\left(\frac{\pi}{2} + n\pi, (-1)^n\right) \middle| n \text{ is an integer}\right\}$
(b)



61. $f^{(n)}(x) = \begin{cases} (-1)^{\frac{n}{2}} \sin x & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \cos x & \text{if } n \text{ is odd} \end{cases}$

62. $f^{(n)}(\theta) = \begin{cases} (-1)^{\frac{n}{2}} \cos\theta & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \sin\theta & \text{if } n \text{ is odd} \end{cases}$

Answers continue on p. 214

63. -1 **64.** -1

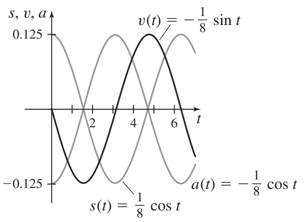
65. (a) $v(t) = -\frac{1}{8}\sin t$ m/s

(b) $t = \frac{\pi}{2} + n\pi$ seconds, n an integer.

(c) $a(t) = -\frac{1}{8}\cos t$ m/s^2

(d) $t = \frac{\pi}{2} + n\pi$ s, n an integer.

(e)



66. (a) From 0 to 5 cm.

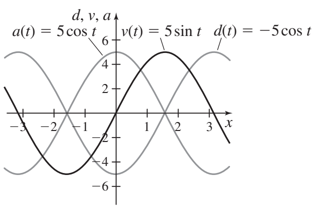
(b) $v(t) = 5\sin t$ cm/s

(c) $t = \frac{\pi}{2}$ s and $t = \frac{3\pi}{3}$ s

(d) $a(t) = 5\cos t$ cm/s^2

(e) $t = \frac{\pi}{2}$ s and $t = \frac{3\pi}{3}$ s

(f)



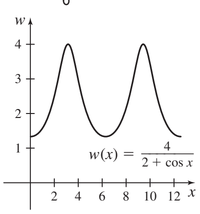
67. (a) $\frac{dy}{d\theta} = -8\sin\theta$ ft/radian

(b) $\frac{dy}{d\theta} = -4\sqrt{2}$ ft/radian

68. (a) $\frac{dA}{d\theta} = \cos 2\theta$ cm^2/radian

(b) $\left.\frac{dA}{d\theta}\right|_{\theta=\frac{x}{6}} = \frac{1}{2}$ cm^2/radian

69. (a)



(b) Max $= 4$, Min $= \frac{4}{3}$

(c) $x = \pi, 2\pi, 3\pi$

(d) $w'(\pi - 0.1) \approx 0.395$,

$w'(2\pi - 0.1) \approx -0.045$

(e) A symmetric wave would give slopes with equal magnitudes but opposite signs.

(a) As you rotate a polarizing filter, θ changes. Find the rate of change of the light intensity I with respect to θ.

(b) Find both the intensity $I(\theta)$ and the rate of change of the intensity with respect to θ, for the angles $\theta = 0°, 45°$, and $90°$. (Remember to use radians for θ.)

73. If $y = \sin x$ and $y^{(n)}$ is the nth derivative of y with respect to x, find the smallest positive integer n for which $y^{(n)} = y$.

74. Use the identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$, with $A = x + h$ and $B = x$, to prove that

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x$$

75. Use the definition of a derivative to prove $\frac{d}{dx}\cos x = -\sin x$.

76. Derivative of $y = \sec x$ Use a derivative rule to show that

$$\frac{d}{dx}\sec x = \sec x \tan x$$

77. Derivative of $y = \csc x$ Use a derivative rule to show that

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

78. Derivative of $y = \cot x$ Use a derivative rule to show that

$$\frac{d}{dx}\cot x = -\csc^2 x$$

79. Let $f(x) = \cos x$. Show that finding $f'(0)$ is the same as finding $\lim_{x\to 0}\frac{\cos x - 1}{x}$.

80. Let $f(x) = \sin x$. Show that finding $f'(0)$ is the same as finding $\lim_{x\to 0}\frac{\sin x}{x}$.

81. If $y = A\sin t + B\cos t$, where A and B are constants, show that $y'' + y = 0$.

Challenge Problem

82. For a differentiable function f, let f^* be the function defined by

$$f^*(x) = \lim_{h\to 0}\frac{f(x+h) - f(x-h)}{h}$$

(a) Find $f^*(x)$ for $f(x) = x^2 + x$.

(b) Find $f^*(x)$ for $f(x) = \cos x$.

(c) Write an equation that expresses the relationship between the functions f^* and f', where f' denotes the derivative of f. Justify your answer.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 208 **1.** If $y = x\sin x$, then $\frac{dy}{dx} =$

(A) $x\cos x + \sin x$ (B) $x\cos x - \sin x$
(C) $\cos x + \sin x$ (D) $(x+1)\cos x$

PAGE 209 **2.** What is $\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3} + h\right) - \cos\frac{\pi}{3}}{h}$?

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $-\frac{\sqrt{3}}{2}$

PAGE 210 **3.** If $f(x) = \tan x$, then $f'\left(\frac{\pi}{3}\right)$ equals

(A) $2\sqrt{3}$ (B) 4 (C) 2 (D) $\frac{1}{4}$

PAGE 210 **4.** The position s (in meters) of an object moving along a horizontal line at time t, $0 \le t \le \frac{\pi}{2}$, (in seconds) is given by $s(t) = 6\sin t + \frac{3}{2}t^2 + 8$. What is the velocity of the object when its acceleration is zero?

(A) 6 m/s (B) $3 + \pi$ m/s
(C) $\frac{6\sqrt{3} + \pi}{2}$ m/s (D) $\left(3\sqrt{3} - \frac{\pi}{2}\right)$ m/s

PAGE 208 **5.** If $y = \sin x$, then $\frac{d^{50}}{dx^{50}}\sin x$ equals

(A) $\sin x$ (B) $-\sin x$ (C) $\cos x$ (D) $-\cos x$

PAGE 209 **6.** If $f(x) = \frac{x}{\cos x}$, find $f'\left(\frac{\pi}{3}\right)$.

(A) $2 - \frac{2\sqrt{3}}{3}\pi$ (B) $1 + \frac{\sqrt{3}}{3}\pi$
(C) $1 - \frac{\sqrt{3}}{3}\pi$ (D) $2 + \frac{2\sqrt{3}}{3}\pi$

PAGE 210 **7.** If $y = x - \tan x$, then $\frac{dy}{dx}$ equals

(A) $1 - \sec x\tan x$ (B) $-\tan^2 x$
(C) $\tan^2 x$ (D) $-\sec^2 x$

PAGE 209 **8.** If $g(x) = e^x\cos x + 2\pi$, then $g'(x) =$

(A) $e^x - \sin x$ (B) $e^x\cos x - e^x\sin x + 3\pi$
(C) $e^x\cos x - e^x\sin x$ (D) $e^x\cos x + e^x\sin x$

PAGE 209 **9.** At which of the following numbers x, $0 \le x \le 2\pi$, does the graph of $y = x + \cos x$ have a horizontal tangent line?

(A) 0 only (B) $\frac{\pi}{2}$ only
(C) $\frac{3\pi}{2}$ only (D) 0 and $\frac{\pi}{2}$ only

PAGE 208 **10.** An equation of the tangent line to the graph of $f(x) = \sin x$ at $x = \frac{2\pi}{3}$ is

(A) $3x + 6y = 4\pi - 3\sqrt{3}$ (B) $3x + 6y = 2\pi + 3\sqrt{3}$
(C) $6y - 3x = 2\pi - 3\sqrt{3}$ (D) $6y - 3x = 4\pi - 3\sqrt{3}$

70. (a) $\left.\frac{dT}{d\theta}\right|_{\theta-0_{max}} = W\sec\theta_{max}\tan\theta_{max}$

(b) $\left.\frac{dT}{d\theta}\right|_{\theta-0} = 0$

(c) $T(0) = W$

71. (a) For November 1, 2018, the projected revenue is \$10,000.00.

For March 1, 2019, the projected revenue is \$25,093.00.

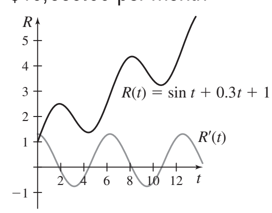
For September 1, 2019, the projected revenue is \$15,411.00.

For January 1, 2020, the projected revenue is \$37,570.00.

(b) $R'(t) = \cos t + 0.3$

(c) \$10,539.00 per month

(d)



(e) Yes.

Answers continue on p. AA-4

CHAPTER 2 PROJECT The *Apollo* Lunar Module


Rolls Press/Popperfoto/Getty Images

The Lunar Module (LM) was a small spacecraft that detached from the *Apollo* Command Module and was designed to land on the Moon. Fast and accurate computations were needed to bring the LM from an orbiting speed of about 5500 ft/s to a speed slow enough to land it within a few feet of a designated target on the Moon's surface. The LM carried a 70-lb computer to assist in guiding it successfully to its target. The approach to the target was split into three phases, each of which followed a reference trajectory specified by NASA engineers.*

The position and velocity of the LM were monitored by sensors that tracked its deviation from the preassigned path at each moment. Whenever the LM strayed from the reference trajectory, control thrusters were fired to reposition it. In other words, the LM's position and velocity were adjusted by changing its acceleration.

The reference trajectory for each phase was specified by the engineers to have the form

$$r_{\text{ref}}(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_T t^3 + \frac{1}{24}S_T t^4 \qquad (1)$$

The variable r_{ref} represents the intended position of the LM at time t before the end of the landing phase. The engineers specified the end of the landing phase to take place at $t = 0$, so that during the phase, t was always negative. Note that the LM was landing in three dimensions, so there were actually three equations like (1). Since each of those equations had this same form, we will work in one dimension, assuming, for example, that r represents the distance of the LM above the surface of the Moon.

1. If the LM follows the reference trajectory, what is the reference velocity $v_{\text{ref}}(t)$?
2. What is the reference acceleration a (ref$t$)?
3. The rate of change of acceleration is called **jerk**. Find the reference jerk $J_{\text{ref}}(t)$.
4. The rate of change of jerk is called **snap**. Find the reference snap $S_{\text{ref}}(t)$.
5. Evaluate $r_{\text{ref}}(t)$, $v_{\text{ref}}(t)$, $a_{\text{ref}}(t)$, $J_{\text{ref}}(t)$, and $S_{\text{ref}}(t)$ when $t = 0$.

The reference trajectory given in equation (1) is a fourth-degree polynomial, the lowest degree polynomial that has enough free parameters to satisfy all the mission criteria. Now we see that the parameters $R_T = r_{\text{ref}}(0)$, $V_T = v_{\text{ref}}(0)$, $A_T = a_{\text{ref}}(0)$, $J_T = J_{\text{ref}}(0)$, and $S_T = S_{\text{ref}}(0)$. The five parameters in equation (1) are referred to as the **target parameters** since they provide the path the LM should follow.

But small variations in propulsion, mass, and countless other variables cause the LM to deviate from the predetermined path. To correct the LM's position and velocity, NASA engineers apply a force to the LM using rocket thrusters. That is, they changed the acceleration. (Remember Newton's second law, $F = ma$.) Engineers modeled the actual trajectory of the LM by

$$r(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_A t^3 + \frac{1}{24}S_A t^4 \qquad (2)$$

We know the target parameters for position, velocity, and acceleration. We need to find the actual parameters for jerk and snap to know the proper force (acceleration) to apply.

6. Find the actual velocity $v = v(t)$ of the LM.
7. Find the actual acceleration $a = a(t)$ of the LM.
8. Use equation (2) and the actual velocity found in Problem 6 to express J_A and S_A in terms of R_T, V_T, A_T, $r(t)$, and $v(t)$.
9. Use the results of Problems 7 and 8 to express the actual acceleration $a = a(t)$ in terms of R_T, V_T, A_T, $r(t)$, and $v(t)$.

The result found in Problem 9 provides the acceleration (force) required to keep the LM in its reference trajectory.

10. When riding in an elevator, the sensation one feels just before the elevator stops at a floor is jerk. Would you want jerk to be small or large in an elevator? Explain. Would you want jerk to be small or large on a roller coaster ride? Explain. How would you explain snap?

*A. R. Klumpp, "*Apollo* Lunar-Descent Guidance," MIT Charles Stark Draper Laboratory, R-695, June 1971, http://www.hq.nasa.gov/alsj/ApolloDescentGuidnce.pdf

TRM **Full Solutions for the Chapter 2 Project**

Chapter Review

THINGS TO KNOW

2.1 Rates of Change and the Derivative

- **Definition** (Form 1) Derivative of a function f at a number c

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists. (p. 167)

Three Interpretations of the Derivative

- *Geometric* If $y = f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$. (p. 167)
- *Rate of change of a function* If $y = f(x)$, the derivative $f'(c)$ is the rate of change of f with respect to x at c. (p. 167)
- *Physical* If the signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s = f(t)$, the derivative $f'(t_0)$ is the velocity of the object at time t_0. (p. 167)

2.2 The Derivative as a Function

- **Definition of a derivative function** (Form 2)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. (p. 172)
- **Theorem** If a function f has a derivative at a number c, then f is continuous at c. (p. 177)
- **Corollary** If a function f is discontinuous at a number c, then f has no derivative at c. (p. 177)

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

- **Leibniz notation** $\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$ (p. 183)
- **Basic derivatives**

$\frac{d}{dx}A = 0$ A is a constant (p. 184) $\quad \frac{d}{dx}x = 1$ (p. 184)

$\frac{d}{dx}e^x = e^x$ (p. 190) $\quad \frac{d}{dx}\ln x = \frac{1}{x}$ (p. 190)

- *Simple Power Rule* $\frac{d}{dx}x^n = nx^{n-1}$, $n \geq 1$, an integer (p. 185)

Properties of Derivatives

- *Sum Rule* (pp. 186, 187) $\frac{d}{dx}[f+g] = \frac{d}{dx}f + \frac{d}{dx}g$
  $(f+g)' = f' + g'$
- *Difference Rule* (p. 187) $\frac{d}{dx}[f-g] = \frac{d}{dx}f - \frac{d}{dx}g$
  $(f-g)' = f' - g'$
- *Constant Multiple Rule* (p. 186) If k is a constant,
  $\frac{d}{dx}[kf] = k\frac{d}{dx}f$
  $(kf)' = k \cdot f'$

2.4 Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

Properties of Derivatives

- *Product Rule* (p. 195) $\frac{d}{dx}(fg) = f\left(\frac{d}{dx}g\right) + \left(\frac{d}{dx}f\right)g$
  $(fg)' = fg' + f'g$
- *Quotient Rule* (p. 196) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\left(\frac{d}{dx}f\right)g - f\left(\frac{d}{dx}g\right)}{g^2}$
  $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
  provided $g(x) \neq 0$
- *Reciprocal Rule* (p. 197) $\frac{d}{dx}\left(\frac{1}{g}\right) = -\frac{\frac{d}{dx}g}{g^2}$
  $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
  provided $g(x) \neq 0$
- *Power Rule* $\frac{d}{dx}x^n = nx^{n-1}$, n an integer (p. 198)
- *Higher-order derivatives* See Table 3 (p. 199)
- *Position Function* $s = s(t)$ (p. 200)
- *Velocity* $v = v(t) = \frac{ds}{dt}$ (p. 200)
- *Acceleration* $a = a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ (p. 200)

2.5 The Derivative of the Trigonometric Functions

Basic Derivatives

$\frac{d}{dx}\sin x = \cos x$ (p. 207) $\quad \frac{d}{dx}\sec x = \sec x \tan x$ (p. 210)

$\frac{d}{dx}\cos x = -\sin x$ (p. 208) $\quad \frac{d}{dx}\csc x = -\csc x \cot x$ (p. 210)

$\frac{d}{dx}\tan x = \sec^2 x$ (p. 210) $\quad \frac{d}{dx}\cot x = -\csc^2 x$ (p. 210)

OBJECTIVES

Section	You should be able to ...	Examples	Review Exercises	Preparing for the AP® Exam: AP® Review Problems
2.1	1 Find equations for the tangent line and the normal line to the graph of a function (p. 162)	1	67–70	
	2 Find the rate of change of a function (p. 163)	2, 3	1, 2, 73 (a)	
	3 Find average velocity and instantaneous velocity (p. 164)	4, 5	71(a), (b); 72(a), (b)	
	4 Find the derivative of a function at a number (p. 166)	6–8	3–8, 75	5
2.2	1 Define the derivative function (p. 171)	1–3	9–12, 77	2
	2 Graph the derivative function (p. 173)	4, 5	9–12, 15–18	
	3 Identify where a function is not differentiable (p. 175)	6–10	13, 14, 75	4
2.3	1 Differentiate a constant function (p. 184)	1		
	2 Differentiate a power function (p. 184)	2, 3	19–22	
	3 Differentiate the sum and the difference of two functions (p. 186)	4–6	23–26, 33, 34, 40, 51, 52, 67	
	4 Differentiate the exponential function $y=e^x$ (p. 189)	7	44, 45, 53, 54, 56, 59, 69	7
2.4	1 Differentiate the product of two functions (p. 194)	1, 2	27, 28, 36, 46, 48–50, 53–56, 60	
	2 Differentiate the quotient of two functions (p. 196)	3–6	29–35, 37–43, 47, 57–59, 68, 73, 74	3, 10
	3 Find higher-order derivatives (p. 198)	7, 8	61–66, 71, 72, 76	
	4 Find the acceleration of an object in rectilinear motion (p. 200)	9	71, 72, 76	8
2.5	1 Differentiate trigonometric functions (p. 207)	1–6	49–60, 70	1, 6, 9

REVIEW EXERCISES

In Problems 1 and 2, use a definition of the derivative to find the rate of change of f at the indicated numbers.

1. $f(x)=\sqrt{x}$ at **(a)** $c=1$ **(b)** $c=4$ 1, 2 **(c)** c any positive real number

2. $f(x)=\dfrac{2}{x-1}$ at **(a)** $c=0$ **(b)** $c=2$ **(c)** c any real number, $c\neq 1$

In Problems 3–8, use a definition of the derivative to find the derivative of each function at the given number.

3. $F(x)=2x+5$ at 2 **4.** $f(x)=4x^2+1$ at -1

5. $f(x)=3x^2+5x$ at 0 **6.** $f(x)=\dfrac{3}{x}$ at 1

7. $f(x)=\sqrt{4x+1}$ at 0 **8.** $f(x)=\dfrac{x+1}{2x-3}$ at 1

In Problems 9–12, use a definition of the derivative to find the derivative of each function. Graph f and f′ on the same set of axes.

9. $f(x)=x-6$ **10.** $f(x)=7-3x^2$

11. $f(x)=\dfrac{1}{2x^3}$ **12.** $f(x)=\pi$

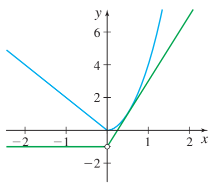
In Problems 13 and 14, determine whether the function f has a derivative at c. If it does, find the derivative. If it does not, explain why. Graph each function.

13. $f(x)=|x^3-1|$ at $c=1$

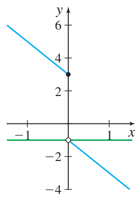
14. $f(x)=\begin{cases}4-3x^2 & \text{if } x\le -1\\ -x^3 & \text{if } x>-1\end{cases}$ at $c=-1$

In Problems 15 and 16, determine whether the graphs represent a function f and its derivative f′. If they do, indicate which is the graph of f and which is the graph of f′.

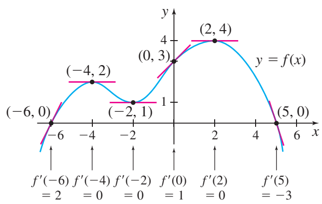
15.



16.



17. Use the information in the graph of $y=f(x)$ to sketch the graph of $y=f'(x)$.



TRM Full Solutions to Chapter 2 Review Exercises

Answers to Chapter 2 Review Exercises

1. (a) $\dfrac{1}{2}$ **(b)** $\dfrac{1}{4}$

(c) $\dfrac{1}{2\sqrt{c}}$

2. (a) -2 **(b)** -2

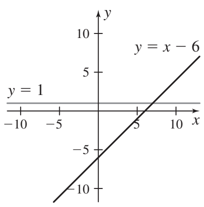
(c) $-\dfrac{2}{(c-1)^2}$

3. 2 **4.** -8

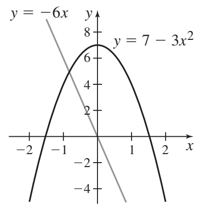
5. 5 **6.** -3

7. 2 **8.** -5

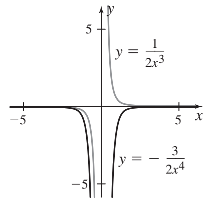
9. $f'(x)=1$



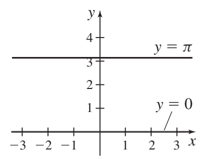
10. $f'(x)=-6x$



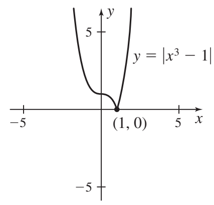
11. $f'(x)=-\dfrac{3}{2x^4}$



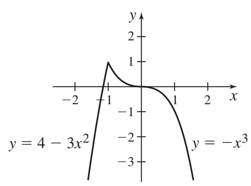
12. $f'(x)=0$



13. No derivative at $c=1$.



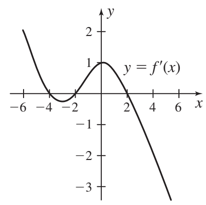
14. No derivative at $c=1$.



15. Not a function and its derivative.

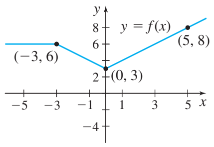
16. Blue is f, green is f'.

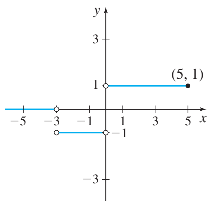
17.



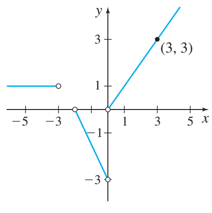
18. A　**19.** $5x^4$
20. $3ax^2$　**21.** x^3
22. $-12x$　**23.** $4x-3$
24. $9x^2+\frac{4}{3}x-5$　**25.** $14x$
26. $\frac{5}{7}$　**27.** $15(x^2-6x+6)$
28. $10x^4-27x^2-5$　**29.** $2+\frac{3}{x^2}$
30. $-\frac{16}{(5x-3)^2}$　**31.** $-\frac{35}{(x-5)^2}$
32. $-\frac{24}{x^{13}}$　**33.** $4x+\frac{10}{x^3}$
34. $-\frac{3}{x^2}-\frac{8}{x^3}$　**35.** $-\frac{a}{x^2}+\frac{3b}{x^4}$
36. $6x^2(x^3-1)$　**37.** $-\frac{6(2x-3)}{(x^2-3x)^3}$
38. $\frac{x^2+2x}{(x+1)^2}$　**39.** $\frac{2t^2(t-3)}{(t-2)^2}$
40. $-\frac{6}{x^3}-\frac{2}{x^2}$　**41.** $-\frac{2z}{(z^2+1)^2}$
42. $\frac{-v^2+2v+1}{(v^2+1)^2}$　**43.** $\frac{1-2z}{(1-z+z^2)^2}$
44. $3e^x+2x$　**45.** $-e^t$
46. $ae^x(2x^2+11+7)$　**47.** $-\frac{x}{e^x}$
48. $8x(x+1)e^{2x}$　**49.** $x\cos x+\sin x$
50. $-2\cos t\sin t$
51. $\sec u(\sec u+\tan u)$
52. $\cos v+\frac{1}{3}\sin v$
53. $e^x(\cos x+\sin x)$
54. $e^x\csc x(1-\cot x)$
55. $2(\cos^2 x-\sin^2 x)$ or $2\cos 2x$
56. $-b\sin x+e^x(\cos x-\sin x)$
57. $2\cos x\sin x$ or $\sin 2x$
58. $\frac{2\csc^2 x}{(1+\cot x)^2}$　**59.** $-\frac{\sin\theta+\cos\theta}{2e^\theta}$
60. 4
61. $f'(x)=50x+30,\ f''(x)=50$
62. $f'(x)=(x+1)e^x,\ f''(x)=(x+2)e^x$
63. $g'(u)=\frac{1}{(2u+1)^2},\ g''(u)=-\frac{4}{(2u+1)^3}$
64. $F'(x)=e^x(3\cos x-\sin x)$,
$F''(x)=2e^x(\cos x-2\sin x)$
65. $f'(u)=-\frac{\sin u+\cos u}{e^u}$,
$f''(u)=\frac{2\sin u}{e^u}$
66. $F'(x)=\frac{x\cos x-\sin x}{x^2}$,
$F''(x)=\frac{2\sin x-x^2\sin x-2x\cos x}{x^3}$

18. Match the graph of $y=f(x)$ with the graph of its derivative.

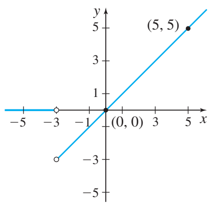




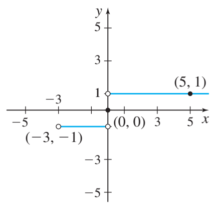
(A)



(B)



(C)



(D)

In Problems 19–60, find the derivative of each function. Treat a and b, if present, as constants.

19. $f(x)=x^5$　**20.** $f(x)=ax^3$
21. $f(x)=\frac{x^4}{4}$　**22.** $f(x)=-6x^2$
23. $f(x)=2x^2-3x$　**24.** $f(x)=3x^3+\frac{2}{3}x^2-5x+7$
25. $F(x)=7(x^2-4)$　**26.** $F(x)=\frac{5(x+6)}{7}$
27. $f(x)=5(x^2-3x)(x-6)$　**28.** $f(x)=(2x^3+x)(x^2-5)$
29. $f(x)=\frac{6x^4-9x^2}{3x^3}$　**30.** $f(x)=\frac{2x+2}{5x-3}$
31. $f(x)=\frac{7x}{x-5}$　**32.** $f(x)=2x^{-12}$
33. $f(x)=2x^2-5x^{-2}$　**34.** $f(x)=2+\frac{3}{x}+\frac{4}{x^2}$
35. $f(x)=\frac{a}{x}-\frac{b}{x^3}$　**36.** $f(x)=(x^3-1)^2$
37. $f(x)=\frac{3}{(x^2-3x)^2}$　**38.** $f(x)=\frac{x^2}{x+1}$
39. $s(t)=\frac{t^3}{t-2}$　**40.** $f(x)=3x^{-2}+2x^{-1}+1$
41. $F(z)=\frac{1}{z^2+1}$　**42.** $f(v)=\frac{v-1}{v^2+1}$
43. $g(z)=\frac{1}{1-z+z^2}$　**44.** $f(x)=3e^x+x^2$
45. $s(t)=1-e^t$　**46.** $f(x)=ae^x(2x^2+7x)$
47. $f(x)=\frac{1+x}{e^x}$　**48.** $f(x)=(2xe^x)^2$
49. $f(x)=x\sin x$　**50.** $s(t)=\cos^2 t$
51. $G(u)=\tan u+\sec u$　**52.** $g(v)=\sin v-\frac{1}{3}\cos v$
53. $f(x)=e^x\sin x$　**54.** $f(x)=e^x\csc x$
55. $f(x)=2\sin x\cos x$　**56.** $f(x)=(e^x+b)\cos x$
57. $f(x)=\frac{\sin x}{\csc x}$　**58.** $f(x)=\frac{1-\cot x}{1+\cot x}$
59. $f(\theta)=\frac{\cos\theta}{2e^\theta}$　**60.** $f(\theta)=4\theta\cot\theta\tan\theta$

In Problems 61–66, find the first derivative and the second derivative of each function.

61. $f(x)=(5x+3)^2$　**62.** $f(x)=xe^x$
63. $g(u)=\frac{u}{2u+1}$　**64.** $F(x)=e^x(\sin x+2\cos x)$
65. $f(u)=\frac{\cos u}{e^u}$　**66.** $F(x)=\frac{\sin x}{x}$

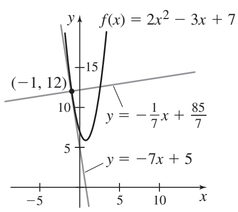
In Problems 67–70, for each function:
(a) *Find an equation of the tangent line to the graph of the function at the indicated point.*
(b) *Find an equation of the normal line to the function at the indicated point.*
(c) *Graph the function, the tangent line, and the normal line on the same screen.*

67. $f(x)=2x^2-3x+7$ at $(-1, 12)$　**68.** $y=\frac{x^2+1}{2x-1}$ at $\left(2, \frac{5}{3}\right)$
69. $f(x)=x^2-e^x$ at $(0, -1)$　**70.** $s(t)=1+2\sin t$ at $(\pi, 1)$

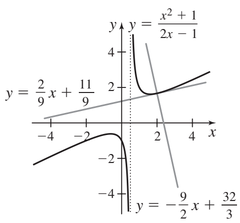
71. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin at time t (in seconds) is given by the position function
$$s=f(t)=t^2-6t$$
(a) Find the average velocity of the object from 0 to 5 s.
(b) Find the velocity at $t=0$, at $t=5$, and at any time t.
(c) Find the acceleration at any time t.

72. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s(t)=t-t^2$, where s is in centimeters and t is in seconds.
(a) Find the average velocity of the object from 1 to 3 s.
(b) Find the velocity of the object at $t=1$ s and $t=3$ s.
(c) What is its acceleration at $t=1$ and $t=3$?

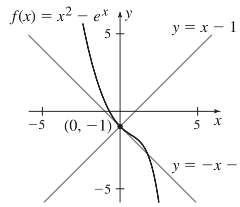
67. (a) $y=-7x+5$　**(b)** $y=\frac{1}{7}x+\frac{85}{7}$
(c)



68. (a) $y=\frac{2}{9}x+\frac{11}{9}$　**(b)** $y=-\frac{9}{2}x+\frac{32}{3}$
(c)



69. (a) $y=-x-1$　**(b)** $y=x-1$
(c)



Answers continue on p. 219

73. Business The price p in dollars per pound when x pounds of a commodity are demanded is modeled by the function

$$p(x) = \frac{10{,}000}{5x + 100} - 5$$

when between 0 and 90 lb are demanded (purchased).

(a) Find the rate of change of price with respect to demand.

(b) What is the revenue function R? (Recall, revenue R equals price times amount purchased.)

(c) What is the marginal revenue R' at $x = 10$ and at $x = 40$ lb?

74. If $f(x) = \frac{x - 1}{x + 1}$ for all $x \neq -1$, find $f'(1)$.

75. If $f(x) = 2 + |x - 3|$ for all x, determine whether the derivative f' exists at $x = 3$.

76. Rectilinear Motion An object in rectilinear motion moves according to the position function $s = 2t^3 - 15t^2 + 24t + 3$, where t is measured in minutes and s in meters.

(a) When is the object at rest?

(b) Find the object's acceleration when $t = 3$.

77. Find the value of the limit below and specify the function f for which this is the derivative.

$$\lim_{\Delta x \to 0} \frac{[4 - 2(x + \Delta x)]^2 - (4 - 2x)^2}{\Delta x}$$

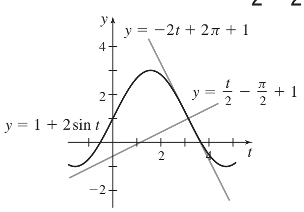
76. (a) $t = 1$ min and $t = 4$ min.

(b) $a(3) = 6$ m/min^2

77. $8x - 16$, $f(x) = (4x - 2)^2$

70. (a) $y = -2t + 2\pi + 1$ **(b)** $y = \frac{t}{2} - \frac{\pi}{2} + 1$

(c)



71. (a) -1 m/s

(b) $v(0) = -6$ m/s, $v(5) = 4$ m/s, $v(t) = 2t - 6$ m/s,

(c) $a(t) = 2$ m/s^2

72. (a) -3 cm/s

(b) $v(1) = -1$ cm/s, $v(3) = -5$ cm/s

(c) $a(1) = -2$ cm/s^2, $a(3) = -2$ cm/s^2

73. (a) $\frac{dp}{dx} = -\frac{50{,}000}{(5x + 100)^2}$ \$/lb^2

(b) $R(x) = \frac{10{,}000x}{5x + 100} - 5x$ dollars

(c) $R'(10) = \frac{355}{9} \approx \$39.44$/lb,

$R'(40) = \frac{55}{9} \approx \$6.11$/lb

74. $f'(1) = \frac{1}{2}$ **75.** $f'(3)$ does not exist.

TRM Full Solutions for Chapter 2 AP® Review Problems

Answers to AP® Review Problems: Chapter 2

1. D
2. D
3. B
4. C
5. A
6. A
7. B
8. D
9. C
10. A
11. C

Preparing for the AP® Exam

AP® REVIEW PROBLEMS: CHAPTER 2

1. If $f(x) = \sec x$, then $f'\left(\frac{\pi}{4}\right) =$

(A) $\frac{\sqrt{2}}{2}$ (B) 2 (C) 1 (D) $\sqrt{2}$

2. If a function f is differentiable at c, then $f'(c)$ is given by

I. $\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$

II. $\lim_{x\to c} \frac{f(x+h) - f(x)}{h}$

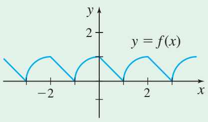
III. $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$

(A) I only (B) III only
(C) I and II only (D) I and III only

3. If $y = \frac{3}{x^2 - 5}$, then $\frac{dy}{dx} =$

(A) $\frac{6x}{(x^2-5)^2}$ (B) $-\frac{6x}{(x^2-5)^2}$
(C) $\frac{6x}{x^2-5}$ (D) $\frac{2x}{(x^2-5)^2}$

4. The graph of the function f is shown below. Which statement about the function is true?



(A) f is differentiable everywhere.
(B) $0 \le f'(x) \le 1$, for all real numbers.
(C) f is continuous everywhere.
(D) f is an even function.

5. The table displays select values of a differentiable function f. What is an approximate value of $f'(2)$?

x	1.996	1.998	2	2.002	2.004
$f(x)$	3.168	3.181	3.194	3.207	3.220

(A) 6.5 (B) 1.154 (C) 0.013 (D) 0.0016

6. If $y = \sin x + xe^x + 6$, what is the instantaneous rate of change of y with respect to x at $x = 5$?

(A) $\cos 5 + 6e^5$ (B) 2
(C) $\cos 5 + 5e^5$ (D) $6e^5 - \cos 5$

7. An equation of the normal line to the graph of $f(x) = 3xe^x + 5$ at $x = 0$ is

(A) $y = 3x + 5$ (B) $y = -\frac{1}{3}x + 5$
(C) $y = \frac{1}{3}x + 5$ (D) $y = -3x + 5$

8. An object moves along a horizontal line so that its position at time t is $s(t) = t^4 - 6t^3 - 2t - 1$. At what time t is the acceleration of the object zero?

(A) at 0 only (B) at 1 only
(C) at 3 only (D) at 0 and 3 only

9. If $f(x) = e^x(\sin x + \cos x)$, then $f'(x) =$

(A) $2e^x(\cos x + \sin x)$ (B) $e^x \cos x$
(C) $2e^x \cos x$ (D) $e^x(\cos^2 x - \sin^2 x)$

10. Find an equation of the tangent line to the graph of $f(x) = \frac{x+3}{x^2+2}$ at $x = 1$.

(A) $5x + 9y = 17$ (B) $9y - 5x = 7$
(C) $5x + 3y = 9$ (D) $5x + 9y = 7$

11. $\lim_{x\to\frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$

(A) 0 (B) −1 (C) 2 (D) Does not exist.

Preparing for the AP® Exam

AP® CUMULATIVE REVIEW PROBLEMS: CHAPTERS 1–2

1. $\lim_{x \to 4} \frac{x-4}{4-x} =$

(A) -4 (B) -1 (C) 0 (D) does not exist

2. $\lim_{x \to 0} \frac{3x + \sin x}{2x} =$

(A) 0 (B) 1 (C) 2 (D) does not exist

3. Let h be defined by

$$h(x) = \begin{cases} f(x) \cdot g(x) & \text{if } x \le 1 \\ k + x & \text{if } x > 1 \end{cases}$$

where f and g are both continuous at all real numbers. If $\lim_{x \to 1} f(x) = 2$ and $\lim_{x \to 1} g(x) = -2$, then for what number k is h continuous?

(A) -5 (B) -4 (C) -2 (D) 2

4. Which function has the horizontal asymptotes $y = 1$ and $y = -1$?

(A) $f(x) = \frac{2}{\pi}\tan^{-1} x$ (B) $f(x) = e^{-x} + 1$

(C) $f(x) = \frac{1-x^2}{1+x^2}$ (D) $f(x) = \frac{2x^2-1}{2x^2+x}$

5. Suppose the function f is continuous at all real numbers and $f(-2) = 1$ and $f(5) = -3$. Suppose the function g is also continuous at all real numbers and $g(x) = f^{-1}(x)$ for all x. The Intermediate Value Theorem guarantees that

(A) $g(c) = 2$ for at least one c between -3 and 1.
(B) $g(c) = 0$ for at least one c between -2 and 5.
(C) $f(c) = 0$ for at least one c between -3 and 1.
(D) $f(c) = 2$ for at least one c between -2 and 5.

6. The line $x = c$ is a vertical asymptote to the graph of the function f. Which of the following statements cannot be true?

(A) $\lim_{x \to c} f(x) = \infty$ (B) $\lim_{x \to \infty} f(x) = c$

(C) $f(c)$ is not defined. (D) f is continuous at $x = c$.

7. The position function of an object moving along a straight line is $s(t) = \frac{1}{15}t^3 - \frac{1}{2}t^2 + 5t^{-1}$. What is the object's acceleration at $t = 5$?

(A) $-\frac{27}{25}$ (B) $-\frac{1}{5}$ (C) $\frac{1}{5}$ (D) $\frac{27}{25}$

8. If the function $f(x) = \begin{cases} 2ax^2 + bx - 1 & \text{if } x \le 3 \\ bx^2 + bx - a & \text{if } x > 3 \end{cases}$

is continuous for all real numbers x, then

(A) $19a - 15b = 1$ (B) $18a - 9b = 1$

(C) $19a - 9b = 1$ (D) $19a + 15b = 1$

9. Find the slope of the tangent line to the graph of $f(x) = xe^x$ at the point $(1, e)$.

(A) 1 (B) e (C) $2e$ (D) e^2

10. An object in rectilinear motion is modeled by the position function

$$s(t) = 3t^4 - 8t^3 - 6t^2 + 24t \qquad t > 0$$

where s is in feet (ft) and t is in seconds (s). Find the acceleration of the object when its velocity is zero.

(A) -24 ft/s², 36 ft/s², and 72 ft/s² only
(B) 36 ft/s² only
(C) 36 ft/s² and 72 ft/s² only
(D) -24 ft/s² and 36 ft/s² only

TRM **Full Solutions for AP® Cumulative Review Problems: Chapters 1–2**

Answers to AP® Cumulative Review Problems: Chapters 1–2

1. B	**2.** C
3. A	**4.** A
5. A	**6.** D
7. D	**8.** C
9. C	**10.** D

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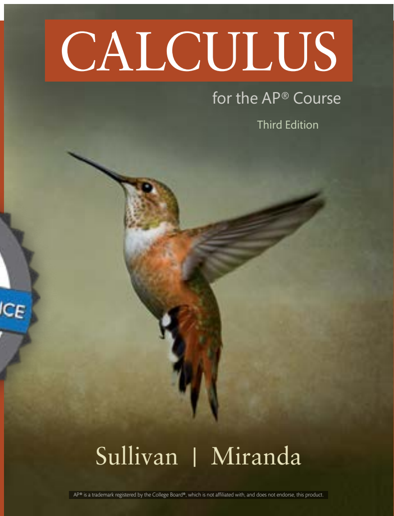

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